

A GENERALIZATION OF CLASSICAL ACTION OF HAMILTONIAN DIFFEOMORPHISMS TO HAMILTONIAN HOMEOMORPHISMS ON FIXED POINTS

JIAN WANG

ABSTRACT. We define boundedness properties on the contractible fixed points set of the time-one map of an identity isotopy on a closed oriented surface with genus $g \geq 1$. In symplectic geometry, a classical object is the notion of action function, defined on the set of contractible fixed points of the time-one map of a Hamiltonian isotopy. We give a dynamical interpretation of this function that permits us to generalize it in the case of a homeomorphism isotopic to identity that preserves a Borel finite measure of rotation vector zero, provided that a boundedness condition is satisfied. We give some properties of the generalized action. In particular, we generalize a result of Schwarz [Sz] about the action function being non-constant which has been proved by using Floer homology. As applications, we generalize some results of Polterovich [P1] about the symplectic and Hamiltonian diffeomorphisms groups on closed oriented surfaces being distortion free, which allows us to give an alternative proof of the C^1 -version of the Zimmer conjecture on closed oriented surfaces.

CONTENTS

0. Introduction	1
1. Notation and Definitions	5
2. Symplectic Action	17
3. Disk Chains	21
4. Extension of the Linking Number	24
5. Boundedness and Existence of the Linking Number	29
6. Action Function	42
7. Application to the group of conservative diffeomorphisms	60
8. Appendix	69
References	72

0. INTRODUCTION

Suppose that (M, ω) is a symplectic manifold with $\pi_2(M) = 0$. Let $I = (F_t)_{t \in [0,1]}$ be a Hamiltonian flow on M with $F_0 = \text{Id}_M$ and $F_1 = F$. Suppose that the function H_t is the Hamiltonian function generating the flow I . Denote by $\text{Fix}_{\text{Cont}, I}(F)$ the set of contractible fixed points of F , that is, x is a fixed point of F and the oriented loop $I(x) : t \mapsto F_t(x)$

Date: September 9, 2012.

defined on $[0, 1]$ can be contractible on M . The classical action function is defined, up to an additive constant, on $\text{Fix}_{\text{Cont}, I}(F)$ as follows (see Section 2.1 for the details)

$$\mathcal{A}_H(x) = \int_{D_x} \omega - \int_0^1 H_t(F_t(x)) dt,$$

where $x \in \text{Fix}_{\text{Cont}, I}(F)$ and $D_x \subset M$ is any 2-simplex with $\partial D_x = I(x)$.

When M is compact, among the properties of F , one may notice the fact that it preserves the volume form $\omega^n = \omega \wedge \cdots \wedge \omega$ and that the “rotation vector” $\rho_{M, I}(\mu) \in H_1(M, \mathbb{R})$ (see Section 1.3) of the finite measure μ induced by ω^n vanishes. In the case of a closed symplectic surface, the fact that a diffeomorphism isotopic to identity preserves a volume form ω whose rotation vector is zero characterizes the fact that it is the time-one map of a 1-periodic Hamiltonian isotopy (see Section 2.1).

The goal of this article is to give a precise dynamical explanation of the action function defined on the set of contractible fixed points in the case of surface. Through defining a weak boundedness property (for example, F is a diffeomorphism or the set $\text{Fix}_{\text{Cont}, I}(F)$ is finite), written WB-property for short, which is a certain boundedness condition about linking numbers of contractible fixed points (see Section 1.5), we define a new action function with the following desired properties and prove that it is a generalization of the classical function:

- It can be naturally generalized for any diffeomorphism (not necessarily C^1) isotopic to the identity that preserves a finite Borel measure of rotation vector zero with no atoms on the contractible fixed points set.
- It can be naturally generalized for any homeomorphism isotopic to the identity that preserves a finite Borel measure of rotation vector zero with total support and no atoms on the contractible fixed points set, provided that the WB-property is satisfied.
- It can be naturally generalized for any homeomorphism isotopic to the identity that preserves a finite ergodic Borel measure μ of rotation vector zero with no atoms on the contractible fixed points set, provided that the WB-property is satisfied.

In addition, we investigate some properties of the new action function: such as the boundedness (Proposition 6.7) and the continuity (Proposition 6.8). Interestingly, we furthermore prove that, in the closed oriented surface case, the new action function is not constant when the measure has total support, which has been proved in [Sz] by Floer homology for the case where the isotopy is a Hamiltonian flow. In [Sz], the time-one map F requires to be at least C^1 -smooth and the contractible fixed point set of F to satisfy certain non-degeneracy. In contrast, we only demand the isotopy to satisfy a much weaker property, the proposed WB-property.

Moreover, we are in particular interested in the conservative diffeomorphism groups of closed oriented surfaces. By applying the new action function to the groups of conservative diffeomorphisms, we generalize some results of Polterovich [P1] about the absence of distortion in the symplectic and Hamiltonian diffeomorphisms groups on closed oriented surface. We proved that the C^1 -conservative diffeomorphism groups have no distortion on closed oriented surfaces, which links to Zimmer conjecture on closed oriented surfaces.

The main results of this article can be summarized as follows.

Let M be a smooth manifold (with boundary or not) and let F be a homeomorphism on M . Denote by $\text{Diff}(M)$ (resp. $\text{Diff}^1(M)$) the group of all diffeomorphisms (resp. C^1 -diffeomorphisms) on M and by $\mathcal{M}(F)$ the set of Borel finite measures on M whose elements are invariant by F . We say that an isotopy $I = (F_t)_{t \in [0,1]}$ on M is an *identity isotopy* if $F_0 = \text{Id}_M$. Our main results are following:

Theorem 0.1. *Let F be the time-one map of an identity isotopy I on a closed oriented surface M with $g \geq 1$. Suppose that $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont},I}(F)$ and that $\rho_{M,I}(\mu) = 0$. In all of the following cases*

- $F \in \text{Diff}(M)$ (not necessarily C^1);
- I satisfies the WB-property, the measure μ has total support;
- I satisfies the WB-property, the measure μ is ergodic,

an action function can be defined which generalizes the classical case.

On closed oriented surfaces, we get the following Proposition 0.2 and Corollary 6.11 that are generalizations of Lemma 2.8 that is proved in [Sz] by using Floer homology.

Proposition 0.2. *Let F be the time-one map of an identity isotopy I on a closed oriented surface M with $g \geq 1$. Suppose that $\mu \in \mathcal{M}(F)$ has total support, no atoms on $\text{Fix}_{\text{Cont},I}(F)$, and $\rho_{M,I}(\mu) = 0$. If I satisfies the WB-property and $F \in \text{Homeo}_*(M) \setminus \{\text{Id}_M\}$ where $\text{Homeo}_*(M)$ is the group of all homeomorphisms isotopic to Id_M on M , then the action function we defined in Theorem 0.1 is not constant.*

Let \tilde{M} be the universal covering space and let \tilde{F} be the time-one map of the lifted identity isotopy of I to \tilde{M} . In fact, in order to generalize the classical action of F , we first define the action function of F on the fixed point set of \tilde{F} (see Section 6.1). And then, we define the action spectrum $\sigma(\tilde{F})$ which is the range of the action function of F (whose domain is the fixed points set of \tilde{F}), and define the action width $\text{width}(\tilde{F}) = \sup_{x,y \in \sigma(\tilde{F})} |x - y|$ (it may be infinite, see Section 6.3). Base on Proposition 0.2, we can get the following Corollary 0.3, Proposition 0.4 and Proposition 0.5 which are generalizations of Theorem 2.1. C, Proposition 2.6. A in [P1].

Corollary 0.3. *Let F be the time-one map of an identity isotopy I on a closed oriented surface M with $g > 1$. If I satisfies the WB-property and $F \in \text{Homeo}_*(M) \setminus \{\text{Id}_M\}$, $\mu \in \mathcal{M}(F)$ has total support and no atoms on $\text{Fix}_{\text{Cont},I}(F)$, then $\sharp\sigma(\tilde{F}) \geq 2$.*

We remark here that Proposition 0.2 and Corollary 0.3 are not valid when the measure has not total support as shown by Example 6.12 and Example 6.13.

We extend the identity isotopy $I = (F_t)_{t \in [0,1]}$ to \mathbb{R} by writing $F_{t+1} = F_t \circ F_1$. We have the following conclusions:

Proposition 0.4. *Under the same hypotheses as Proposition 0.2, there exists a constant $C > 0$ such that $\text{width}(\tilde{F}^n) \geq C \cdot n$ for every $n \geq 1$ where \tilde{F}^n is the time-one map of the lifted identity isotopy of $I^n = (F_t)_{t \in [0,n]}$ to \tilde{M} .*

Proposition 0.5. *Under the same hypotheses as Corollary 0.3, there exists a constant $C > 0$ such that $\text{width}(\tilde{F}^n) \geq C \cdot n$ for every $n \geq 1$.*

Fix a Borel finite measure μ whose support is the whole space. Denote by $\text{Homeo}_*(M, \mu)$ the subgroup of $\text{Homeo}_*(M)$ whose element preserves the measure μ , and by $\text{Hameo}(M, \mu)$ the subgroup of $\text{Homeo}_*(M, \mu)$ whose elements satisfy furthermore that $\rho_{M,I}(\mu) = 0$. For convenience, we write M_g the closed oriented surface with the genus $g \geq 1$.

Based on the previous result listed above, we can study the periodic homeomorphisms of surfaces. Applying Proposition 0.4, Proposition 0.5 and a result of Fathi [Fa](see Section 7.1), we can get the following corollary :

Corollary 0.6. *The groups $\text{Hameo}(\mathbb{T}^2, \mu)$, $\text{Homeo}_*(M_g)$ ($g > 1$) are torsion free.*

Let us now recall the definition of distortion (see [P1]). If \mathcal{G} is a finitely generated group with generators $\{g_1, \dots, g_s\}$, then $f \in \mathcal{G}$ is said to be a *distortion element* of \mathcal{G} provided that f has infinite order and

$$\liminf_{n \rightarrow +\infty} \frac{\|f^n\|_{\mathcal{G}}}{n} = 0,$$

where $\|f^n\|_{\mathcal{G}}$ is the word length of f^n in the generators $\{g_1, \dots, g_s\}$. If \mathcal{G} is not finitely generated, then we say that $f \in \mathcal{G}$ is distorted in \mathcal{G} if it is distorted in some finitely generated subgroup of \mathcal{G} .

Given two positive sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n \succeq b_n$ if there is $c > 0$ such that $a_n \geq cb_n$ for all $n \in \mathbb{N}$, and $a_n \sim b_n$ if $a_n \succeq b_n$ and $a_n \preceq b_n$.

Denote by $\text{Ham}^1(M, \mu)$ the group

$$\text{Hameo}(M, \mu) \cap \text{Diff}^1(M)$$

and by $\text{Diff}_*^1(M, \mu)$ the group

$$\text{Homeo}_*(M, \mu) \cap \text{Diff}^1(M).$$

Finally, we study the existence of distortion of the groups $\text{Ham}^1(\mathbb{T}^2, \mu)$ and $\text{Diff}_*^1(M_g, \mu)$ and get the following result:

Theorem 0.7. *Let $F \in \text{Diff}_*^1(M_g, \mu) \setminus \{\text{Id}_{M_g}\}$ ($g > 1$) (resp. $F \in \text{Ham}^1(\mathbb{T}^2, \mu) \setminus \{\text{Id}_{\mathbb{T}^2}\}$), and $\mathcal{G} \subset \text{Diff}_*^1(M_g, \mu)$ ($g > 1$) (resp. $\mathcal{G} \subset \text{Ham}^1(\mathbb{T}^2, \mu)$) be a finitely generated subgroup containing F , then*

$$\|F^n\|_{\mathcal{G}} \sim n.$$

As a consequence, the groups $\text{Diff}_^1(M_g, \mu)$ ($g > 1$) and $\text{Ham}^1(\mathbb{T}^2, \mu)$ have no distortion.*

Applying Theorem 0.7, some algebraic properties of the lattice $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) and mapping class group (see Section 7.2 for the details), we get the following theorems.

Theorem 0.8. *Every homomorphism from $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) to $\text{Ham}^1(\mathbb{T}^2, \mu)$ or $\text{Diff}_*^1(M_g, \mu)$ ($g > 1$) is trivial.*

Theorem 0.9. *Every homomorphism from $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) to $\text{Diff}^1(M_g, \mu)$ ($g > 1$) has only finite images.*

Remark here that Theorem 0.9 is a more general conjecture of Zimmer (see [Z, P2, FH2]) in the special surfaces case. The reader can find more information about Zimmer conjecture in Section 7.2.

The article is organized as follows. In Section 1, we will first introduce some notations and recall the precise definitions of some important mathematical objects. And then we will define the linking number on contractible fixed points and the boundedness properties. Finally, we will study some conditions for these properties to hold. In Section 2, we will recall the classical action function in symplectic geometry and analyze how to generalize the action function to a more general case on closed oriented surfaces. In the end of this section, our main theorem is stated. In Section 3, we will recall some well known results about the plan and the open annulus, and extend some results of Franks to serve as the technical preliminaries of this article. In Section 4, we will first extend the definition of the linking number defined in Section 1 to positively recurrent points, which is one of the main ingredients of this article, and then we will give some elementary properties of the extended linking number. In Section 5, we will first study the boundedness of the extended linking number when it exists, and then study the existence and the boundedness of the linking number in the conservative case. In Section 6, based on the extended linking number and its properties studied in Sections 4 and 5, we will define a new action function and prove that it is a generalization of the classical one, which is our main theorem. We study the properties of the new action function, including boundedness and continuity. Furthermore, we prove that, on the closed oriented surface case, the new action function is not constant when the measure has total support whose smooth case has been proved in [Sz] by using Floer homology. We also give examples to illuminate that this result is not true any more when the measure has not total support. In Section 7, by applying our generalized action function to the groups of conservative diffeomorphisms, we generalize the results of Polterovich about the absence of distortion in the symplectic and Hamiltonian diffeomorphisms groups on closed oriented surface in [P1]. We proved that the C^1 -conservative diffeomorphism groups have no distortion on closed oriented surfaces, which permits us to give an alternative proof of the C^1 -version of the Zimmer conjecture on oriented closed surfaces when the measure is a Borel finite measure with full support. In Section 8, we provide a proof of a well known fact required in this article but which we have not found in the literature. In addition, we construct some examples to further complete our results.

Acknowledgements. I would like to thank my advisors, Patrice Le Calvez and Zhiying Wen, for many helpful discussions and suggestions. I also thank Olivier Jaulent and Juliana Xavier for their explanations of their results to me. I am grateful to Bassam Fayad and Lucien Guillou for careful reading the manuscript and many useful remarks.

1. NOTATION AND DEFINITIONS

We denote by $|\cdot|$ the usual Euclidean metric on \mathbb{R}^k and by $\mathbf{S}^{k-1} = \{x \in \mathbb{R}^k \mid |x| = 1\}$ the unit sphere.

If A is a set, we write $\sharp A$ for the cardinality of A . If G is a group and e is its unit element, we write $G^* = G \setminus \{e\}$. If (S, σ, μ) is a measured space and V is any finite dimensional linear space, denote $L^1(S, V, \mu)$ the set of μ -integrable functions from S to V . If X is a topological space and A is a subset of X , denote respectively $\text{Int}(A)$, $\text{Cl}(A)$ and ∂A the interior, the closure and the boundary of A .

If M is a smooth manifold (with boundary or not), we denote by $\text{Homeo}(M)$ the set of all homeomorphisms of M .

1.1. Identity isotopies. An *identity isotopy* $I = (F_t)_{t \in [0,1]}$ on M is a continuous path

$$\begin{aligned} [0, 1] &\rightarrow \text{Homeo}(M) \\ t &\mapsto F_t \end{aligned}$$

such that $F_0 = \text{Id}_M$, the last set being endowed with the compact-open topology. We naturally extend this map to \mathbb{R} by writing $F_{t+1} = F_t \circ F_1$. We can also define the inverse isotopy of I as $I^{-1} = (F_{-t})_{t \in [0,1]} = (F_{1-t} \circ F_1^{-1})_{t \in [0,1]}$. We denote by $\text{Homeo}_*(M)$ (resp. $\text{Diff}_*(M)$, $\text{Diff}_*^1(M)$) the set of all homeomorphisms (resp. diffeomorphisms, C^1 -diffeomorphisms) of M that are isotopic to the identity.

A *path* on a manifold M is a continuous map $\gamma : J \rightarrow M$ defined on a nontrivial interval J (up to an increasing reparametrization). We can talk of a proper path (i.e. $\gamma^{-1}(K)$ is compact for any compact set K) or a compact path (i.e. J is compact). When γ is a compact path, $\gamma(\inf J)$ and $\gamma(\sup J)$ are the *ends* of γ . We say that a compact path γ is a *loop* if the two ends of γ coincide. The inverse of the path γ is defined by $\gamma^{-1} : t \mapsto \gamma(-t)$, $t \in -J$. If $\gamma_1 : J_1 \rightarrow M$ and $\gamma_2 : J_2 \rightarrow M$ are two paths such that

$$b_1 = \sup J_1 \in J_1, \quad a_2 = \inf J_2 \in J_2 \quad \text{and} \quad \gamma_1(b_1) = \gamma_2(a_2),$$

then the *concatenation* γ_1 and γ_2 is defined on $J = J_1 \cup (J_2 + (b_1 - a_2))$ in the classical way, where $(J_2 + (b_1 - a_2))$ represents the translation of J_2 by $(b_1 - a_2)$:

$$\gamma_1 \gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } t \in J_1; \\ \gamma_2(t + a_2 - b_1) & \text{if } t \in J_2 + (b_1 - a_2). \end{cases}$$

Let \mathcal{I} be an interval (maybe infinite) of \mathbb{Z} . If $\{\gamma_i : J_i \rightarrow M\}_{i \in \mathcal{I}}$ is a family of compact paths satisfying that $\gamma_i(\sup(J_i)) = \gamma_{i+1}(\inf(J_{i+1}))$ for every $i \in \mathcal{I}$, then we can define their concatenation $\prod_{i \in \mathcal{I}} \gamma_i$.

If $\{\gamma_i\}_{i \in \mathcal{I}}$ is a family of compact paths where $\mathcal{I} = \bigsqcup_{j \in \mathcal{J}} \mathcal{I}_j$ and \mathcal{I}_j is an interval of \mathbb{Z} such that $\prod_{i \in \mathcal{I}_j} \gamma_i$ is well defined (in the concatenation sense) for every $j \in \mathcal{J}$, we define their *product* by abusing notations:

$$\prod_{i \in \mathcal{I}} \gamma_i = \prod_{j \in \mathcal{J}} \prod_{i \in \mathcal{I}_j} \gamma_i.$$

The *trajectory* of a point z for the isotopy $I = (F_t)_{t \in [0,1]}$ is the oriented path $I(z) : t \mapsto F_t(z)$ defined on $[0, 1]$. Suppose that $\{I_k\}_{1 \leq k \leq k_0}$ is a family of identity isotopies on M . Write $I_k = (F_{k,t})_{t \in [0,1]}$. We can define a new identity isotopy $I_{k_0} \cdots I_2 I_1 = (F_t)_{t \in [0,1]}$ by concatenation as follows

$$(1.1.1) \quad F_t(z) = F_{k, k_0 t - (k-1)}(F_{k-1,1} \circ F_{k-2,1} \circ \cdots \circ F_{1,1}(z)) \quad \text{if} \quad \frac{k-1}{k_0} \leq t \leq \frac{k}{k_0}.$$

In particular, $I^{k_0}(z) = \prod_{k=0}^{k_0-1} I(F^k(z))$ when $I_k = I$ for all $1 \leq k \leq k_0$.

We write $\text{Fix}(F)$ for the set of fixed points of F . A fixed point z of $F = F_1$ is *contractible* if $I(z)$ is homotopic to zero. We write $\text{Fix}_{\text{Cont}, I}(F)$ for the set of contractible fixed points of F , which obviously depends on I .

1.2. The algebraic intersection number. The choice of an orientation on M permits us to define the algebraic intersection number $\Gamma \wedge \Gamma'$ between two loops. We keep the same notation $\Gamma \wedge \gamma$ for the algebraic intersection number between a loop and a path γ when it is defined, for example, when γ is proper or when γ is compact path whose extremities are not in Γ . Similarly, we write $\gamma \wedge \gamma'$ for the algebraic intersection number of two path γ and γ' when it is defined, for example, when γ and γ' are compact paths and the ends of γ (resp. γ') are not on γ' (resp. γ).

1.3. Rotation vector.

1.3.1. The definition of rotation vector. Let us introduce the classical notion of rotation vector (defined originally in [St]). If Γ is a loop on a smooth manifold M , write $[\Gamma]_M \in H_1(M, \mathbb{Z})$ for the homology class of Γ . Suppose that $F \in \text{Homeo}_*(M)$ is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$. Let $\text{Rec}^+(F)$ be the set of positively recurrent points of F . If $z \in \text{Rec}^+(F)$, fix an open disk $U \subset M$ containing z , and write $\{F^{n_k}(z)\}_{k \geq 1}$ for the subsequence of the positive orbit of z obtained by keeping the points that are in U . For any $k \geq 0$, choose a simple path $\gamma_{F^{n_k}(z), z}$ in U joining $F^{n_k}(z)$ to z . The homology class $[\Gamma_k]_M \in H_1(M, \mathbb{Z})$ of the loop $\Gamma_k = I^{n_k}(z) \gamma_{F^{n_k}(z), z}$ does not depend on the choice of $\gamma_{F^{n_k}(z), z}$. Say that z has a *rotation vector* $\rho_{M, I}(z) \in H_1(M, \mathbb{R})$ if

$$\lim_{l \rightarrow +\infty} \frac{1}{n_{k_l}} [\Gamma_{k_l}]_M = \rho_{M, I}(z)$$

for any subsequence $\{F^{n_{k_l}}(z)\}_{l \geq 1}$ which converges to z . Neither the existence nor the value of the rotation vector depends on the choice of U .

1.3.2. The existence of rotation number in the compact case. Suppose that M is compact and that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on M . Recall that $\mathcal{M}(F)$ is the set of Borel finite measures on M whose elements are invariant by F . If $\mu \in \mathcal{M}(F)$, we can define the rotation vector $\rho_{M, I}(z)$ for μ -almost every positively recurrent point [Lec1]. Let us explain why.

Let U be an open disk of M that is the interior of a closed topological disk. For every couple $(z', z'') \in U^2$, choose a simple path $\gamma_{z', z''}$ in U joining z' to z'' . We can define the first return map $\Phi : \text{Rec}^+(F) \cap U \rightarrow \text{Rec}^+(F) \cap U$ and write $\Phi(z) = F^{\tau(z)}(z)$, where $\tau(z)$ is the first return time, that is, the least number $n \geq 1$ such that $F^n(z) \in U$. By Poincaré Recurrence Theorem, this map is defined μ -almost everywhere on U . For every $z \in \text{Rec}^+(F) \cap U$ and $n \geq 1$, define

$$\tau_n(z) = \sum_{i=0}^{n-1} \tau(\Phi^i(z)), \quad \Gamma_z^n = I^{\tau_n(z)}(z) \gamma_{\Phi^n(z), z}.$$

Observe now that

$$[\Gamma_z^n]_M = \sum_{i=0}^{n-1} [\Gamma_{\Phi^i(z)}^1]_M.$$

By the classical Kac's lemma (see [Ka]), we have

$$\int_U \tau \, d\mu = \mu \left(\bigcup_{k \geq 0} F^k(U) \right) = \mu \left(\bigcup_{k \in \mathbb{Z}} F^k(U) \right).$$

Indeed, we have the following measurable partitions (modulo sets of measure zero):

$$U = \bigsqcup_{i \geq 1} U_i \quad \text{and} \quad \bigcup_{k \geq 0} F^k(U) = \bigsqcup_{i \geq 1} \bigsqcup_{0 \leq j \leq i-1} F^j(U_i),$$

where $U_i = \tau^{-1}(\{i\})$, therefore

$$\mu \left(\bigcup_{k \geq 0} F^k(U) \right) = \sum_{i \geq 1} \sum_{0 \leq j \leq i-1} \mu(U_i) = \sum_{i \geq 1} i \mu(U_i) = \int_U \tau \, d\mu.$$

Hence, we get $\tau \in L^1(U, \mathbb{R}, \mu)$. In the case where M is compact, let us prove that the function $z \mapsto [\Gamma_z^1]_M / \tau(z)$ is bounded on $\text{Rec}^+(F) \cap U$ and hence that the map $z \mapsto [\Gamma_z^1]_M$ belongs to $L^1(U, H_1(M, \mathbb{R}), \mu)$.

Indeed, it is sufficient to prove that for every cohomology class $\kappa \in H^1(M, \mathbb{R})$, there exists a constant K_κ such that $|\langle \kappa, [\Gamma_z^1]_M \rangle| \leq K_\kappa \tau(z)$. Let λ be a closed form that represents κ . The function $g_\lambda : z \mapsto \int_{I(z)} \lambda$ is well defined, since λ is closed, and continuous. It is bounded since M is compact. As $\text{Cl}(U)$ is a closed disk, we can find an open disk U' containing $\text{Cl}(U)$ and a primitive h_λ of λ on U' . This primitive is bounded on $\text{Cl}(U)$. This implies that for every $z \in \text{Rec}^+(F) \cap U$, we have

$$\begin{aligned} (1.3.1) \quad |\langle [\lambda], [\Gamma_z^1]_M \rangle| &= \left| \int_{\Gamma_z^1} \lambda \right| = \left| \sum_{i=0}^{\tau(z)-1} \int_{I(F^i(z))} \lambda + \int_{\lambda_{\Phi(z), z}} \lambda \right| \\ (1.3.2) \quad &\leq \tau(z) \max_{z \in M} |g_\lambda(z)| + 2 \sup_{z \in U} |h_\lambda(z)| \\ (1.3.3) \quad &\leq \tau(z) (\max_{z \in M} |g_\lambda(z)| + 2 \sup_{z \in U} |h_\lambda(z)|). \end{aligned}$$

By Birkhoff Ergodic Theorem, for μ -almost every point on $\text{Rec}^+(F) \cap U$, the sequence $\{\tau_n(z)/n\}_{n \geq 1}$ converges to a real number $\tau^*(z) \geq 1$, and the sequence $\{[\Gamma_z^n]_M/n\}_{n \geq 1}$ converges to $[\Gamma_z^*]_M \in H_1(M, \mathbb{R})$. The positively recurrent points of F in U are exactly the positively recurrent points of Φ because U is open. We deduce that μ -almost every point $z \in \text{Rec}^+(F) \cap U$ has a rotation vector $\rho_{M,I}(z) = [\Gamma_z^*]_M / \tau^*(z)$. Since U is arbitrarily chosen, we deduce that μ -almost every point $z \in \text{Rec}^+(F)$ has a rotation vector. The function $z \mapsto [\Gamma_z^1]_M / \tau(z)$ is bounded on $\text{Rec}^+(F) \cap U$, so is the function

$$\rho_{M,I} : z \mapsto \lim_{n \rightarrow +\infty} \frac{\sum_{i=0}^{n-1} [\Gamma_{\Phi^i(z)}^1]_M}{\sum_{i=0}^{n-1} \tau(\Phi^i(z))}$$

on $\text{Rec}^+(F) \cap U$. As M can be covered by finitely many such open disks, we deduce that $\rho_{M,I}$ is uniformly bounded on $\text{Rec}^+(F)$. Therefore, we can define the *rotation vector of the measure*

$$\rho_{M,I}(\mu) = \int_M \rho_{M,I} d\mu \in H_1(M, \mathbb{R}).$$

1.3.3. The rotation number of an open annulus. Let $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ be the open annulus. Let us denote the covering map

$$\begin{aligned} \pi : \mathbb{R}^2 &\rightarrow \mathbb{A} \\ (x, y) &\mapsto (x + \mathbb{Z}, y), \end{aligned}$$

and T the generator of the covering transformation group

$$\begin{aligned} T : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto (x + 1, y). \end{aligned}$$

When $F \in \text{Homeo}_*(\mathbb{A})$, we have a simple way to define the “rotation vector” given in 1.3.1 if we observe that $H_1(\mathbb{A}, \mathbb{R}) = \mathbb{R}$. We will say that a positively recurrent point z has a *rotation number* $\rho_{\mathbb{A}, \tilde{F}}(z)$ for a lift \tilde{F} of F to the universal cover \mathbb{R}^2 of \mathbb{A} , if for every subsequence $\{F^{n_k}(z)\}_{k \geq 1}$ of $\{F^n(z)\}_{n \geq 1}$ which converges to z , we have

$$\lim_{k \rightarrow +\infty} \frac{p_1 \circ \tilde{F}^{n_k}(\tilde{z}) - p_1(\tilde{z})}{n_k} = \rho_{\mathbb{A}, \tilde{F}}(z)$$

for every $\tilde{z} \in \pi^{-1}(z)$, where $p_1 : (x, y) \mapsto x$ is the first projection. We denote the set of rotation numbers of positively recurrent points of F for \tilde{F} as $\text{Rot}(\tilde{F})$. In particular, the rotation number $\rho_{\mathbb{A}, \tilde{F}}(z)$ always exists when z is a fixed point of F . We denote the set of rotation numbers of fixed points of F as $\text{Rot}_{\text{Fix}(F)}(\tilde{F})$.

It is well known that a positively recurrent point of F is also a positively recurrent point of F^q for all $q \in \mathbb{N}$ (we give a proof in Appendix, see Lemma 8.1). By the definition of rotation number, we easily get that $\text{Rot}(\tilde{F})$ satisfies the following elementary properties.

1. $\text{Rot}(T^k \circ \tilde{F}) = \text{Rot}(\tilde{F}) + k$ for every $k \in \mathbb{Z}$;
2. $\text{Rot}(\tilde{F}^q) = q \text{Rot}(\tilde{F})$ for every $q \geq 1$.

1.4. Linking number of contractible fixed points.

1.4.1. We begin by recalling some results about identity isotopies, which will be often used in the literature.

Remark 1.1. Suppose that M is an oriented compact surface and that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on M . When $z \in \text{Fix}_{\text{Cont}, I}(F)$, there is another identity isotopy $I' = (F'_t)_{t \in [0,1]}$ homotopic to I with fixed endpoints such that I' fixes z (see, for example, [J, Proposition 4.1]), that is, there is a continuous map $H : [0, 1] \times [0, 1] \times M \rightarrow M$ such that

- $H(0, t, z) = F_t(z)$ and $H(1, t, z) = F'_t(z)$ for all $t \in [0, 1]$;
- $H(s, 0, z) = \text{Id}_M(z)$ and $H(s, 1, z) = F(z)$ for all $s \in [0, 1]$;
- $F'_t(z) = z$ for all $t \in [0, 1]$.

Lemma 1.2. *Let \mathbf{S}^2 be the 2-sphere and $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on \mathbf{S}^2 . For every three different fixed points z_i ($i = 1, 2, 3$) of F_1 , there exists another identity isotopy $I' = (F'_t)_{t \in [0,1]}$ from $\text{Id}_{\mathbf{S}^2}$ to F_1 such that I' fixes z_i ($i = 1, 2, 3$).*

Proof. We identify the sphere \mathbf{S}^2 to the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The Möbius transformation $\mathcal{M}(z) = \frac{az+b}{cz+d}$ maps the triple (v_1, v_2, v_3) to the triple $(\omega_1, \omega_2, \omega_3)$ (see Chapter 3 of [N] for a beautifully illustrated introduction to Möbius transformations) where

$$(1.4.1) \quad \begin{aligned} a &= \det \begin{pmatrix} v_1\omega_1 & \omega_1 & 1 \\ v_2\omega_2 & \omega_2 & 1 \\ v_3\omega_3 & \omega_3 & 1 \end{pmatrix} & b &= \det \begin{pmatrix} v_1\omega_1 & v_1 & \omega_1 \\ v_2\omega_2 & v_2 & \omega_2 \\ v_3\omega_3 & v_3 & \omega_3 \end{pmatrix} \\ c &= \det \begin{pmatrix} v_1 & \omega_1 & 1 \\ v_2 & \omega_2 & 1 \\ v_3 & \omega_3 & 1 \end{pmatrix} & d &= \det \begin{pmatrix} v_1\omega_1 & v_1 & 1 \\ v_2\omega_2 & v_2 & 1 \\ v_3\omega_3 & v_3 & 1 \end{pmatrix}. \end{aligned}$$

If one of the points v_i or w_i in 1.4.1 is ∞ , then we first divide all four determinants by this variable and then take the limit as the variable approaches ∞ . Replacing v_i, w_i by $v_i(t) = F_t(z_i)$ and $w_i(t) = z_i$ ($i = 1, 2, 3$ and $t \in [0, 1]$) in the matrices above, we get the matrix functions a_t, b_t, c_t and d_t .

Let

$$\mathcal{M}(t, z) = \frac{a_t z + b_t}{c_t z + d_t}$$

and

$$I'(z)(t) = F'_t(z) = \mathcal{M}(t, F_t(z)).$$

By the construction, I' is an isotopy of \mathbf{S}^2 from $\text{Id}_{\mathbf{S}^2}$ to F_1 that fixes z_i ($i = 1, 2, 3$). \square

As a consequence, we have the following corollary.

Corollary 1.3. *Let $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on \mathbb{C} . For any two different fixed points z_1 and z_2 of F_1 , there exists another identity isotopy I' from $\text{Id}_{\mathbb{C}}$ to F_1 such that I' fixes z_1 and z_2 .*

Remark 1.4. Let $z_i \in \mathbf{S}^2$ ($i = 1, 2, 3$) and $\text{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)$ be the identity component of the space of all homeomorphisms of \mathbf{S}^2 leaving z_i ($i = 1, 2, 3$) pointwise fixed (for the compact-open topology). It is well known that $\pi_1(\text{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)) = 0$ (see [Ham2], [Han]). It implies that any two identity isotopies $I, I' \subset \text{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)$ with fixed endpoints are homotopic. As a consequence, let $\text{Homeo}_*(\mathbb{C}, z_1, z_2)$ be the identity component of the space of all homeomorphisms of \mathbb{C} leaving two different points z_1 and z_2 of \mathbb{C} pointwise fixed, we have $\pi_1(\text{Homeo}_*(\mathbb{C}, z_1, z_2)) = 0$.

1.4.2. Let M be a surface that is homeomorphic to the complex plane \mathbb{C} and $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on M . Let us define the *linking number* $i_I(z, z') \in \mathbb{Z}$ for every two different fixed points z and z' of F_1 . It is the degree of the map $\xi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ defined by

$$\xi(e^{2i\pi t}) = \frac{h \circ F_t(z') - h \circ F_t(z)}{|h \circ F_t(z') - h \circ F_t(z)|},$$

where $h : M \rightarrow \mathbb{C}$ is a homeomorphism. The linking number does not depend on the chosen h .

It is well known that $U(1)$ is a strong deformation retract of $\text{Homeo}_*(\mathbb{C})$ (see [Kn] or [Ler, Theorem 2.9]). Consider the isotopy $R = (r_t)_{t \in [0,1]}$ where $r_t = e^{2i\pi t}$. If $I = (F_t)_{t \in [0,1]}$ is an identity isotopy and $k \in \mathbb{Z}$, we can define the identity isotopy $R^k I$ by concatenation. If $I' = (F'_t)_{t \in [0,1]}$ is another identity isotopy with $F'_1 = F_1$, then there exists a unique integer k such that I' is homotopic to $R^k I$.

Therefore, if $I = (F_t)_{t \in [0,1]}$ and $I' = (F'_t)_{t \in [0,1]}$ are two identity isotopies on M with $F'_1 = F_1$, then there exist $k \in \mathbb{Z}$ such that $i_{I'}(z, z') = i_I(z, z') + k$ for any distinct fixed points z' and z' of F_1 .

1.4.3. Let F be the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on a closed oriented surface M of genus $g \geq 1$ and \tilde{F} be the time-one map of the lifted identity isotopy $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]}$ on the universal cover \tilde{M} of M . When $g > 1$, it is well known that $\pi_1(\text{Homeo}_*(M)) \simeq 0$ ([Ham2]). It implies that any two identity isotopies $I, I' \subset \text{Homeo}_*(M)$ with fixed endpoints are homotopic. Hence, I is unique up to homotopy, it implies that \tilde{F} is uniquely defined and does not depend on the choice of the isotopy from Id_M to F . When $g = 1$, $\pi_1(\text{Homeo}_*(M)) \simeq \mathbb{Z}^2$ (see [Ham1]), \tilde{F} depends on the isotopy I . The universal cover \tilde{M} is homeomorphic to \mathbb{C} .

Let $\pi : \tilde{M} \rightarrow M$ be the covering map and G be the covering transformation group. Denote respectively by Δ and $\tilde{\Delta}$ the diagonal of $\text{Fix}_{\text{Cont}, I}(F) \times \text{Fix}_{\text{Cont}, I}(F)$ and the diagonal of $\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})$. Endow the surface M with a Riemannian metric and denote by d the distance induced by the metric. Lift the Riemannian metric to \tilde{M} and write \tilde{d} for the distance induced by the metric.

We define the *linking number* $i(\tilde{F}; \tilde{z}, \tilde{z}')$ for every pair $(\tilde{z}, \tilde{z}') \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ as

$$(1.4.2) \quad i(\tilde{F}; \tilde{z}, \tilde{z}') = i_{\tilde{F}}(\tilde{z}, \tilde{z}').$$

This is a special case of the linking number that we have defined in 1.4.2.

We give some properties of $i(\tilde{F}; \tilde{z}, \tilde{z}')$ as follows.

- (P1): $i(\tilde{F}; \tilde{z}, \tilde{z}')$ is locally constant on $(\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$;
- (P2): $i(\tilde{F}; \tilde{z}, \tilde{z}')$ is invariant by covering transformation, that is,

$$i(\tilde{F}; \alpha(\tilde{z}), \alpha(\tilde{z}')) = i(\tilde{F}; \tilde{z}, \tilde{z}') \quad \text{for every } \alpha \in G;$$

- (P3): $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$ if $\pi(\tilde{z}) = \pi(\tilde{z}')$;

- (P4): there exists K such that $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$ if $\tilde{d}(\tilde{z}, \tilde{z}') \geq K$.

Indeed, the property P1 is true by continuity. The property P2 is true because the linking number does not change when you replace h by $h \circ \alpha$ (see 1.4.2). By Remark 1.1, we can choose an isotopy I' that is homotopic to I and fixes $\pi(\tilde{z})$, then the lift \tilde{I}' of I' fixes \tilde{z} and \tilde{z}' . Thus the property P3 holds. Finally, let

$$K = \sup\{\tilde{d}(\tilde{F}_t(\tilde{z}), \tilde{F}_{t'}(\tilde{z}')) \mid (t, t', \tilde{z}) \in [0, 1]^2 \times \text{Fix}(\tilde{F})\}.$$

The value K is well defined because $\text{Fix}_{\text{Cont},I}(F) = \pi(\text{Fix}(\tilde{F}))$ is compact and $\tilde{F}_t \circ \alpha = \alpha \circ \tilde{F}_t$ for all $t \in [0, 1]$ and $\alpha \in G$. Obviously, when $\tilde{d}(\tilde{z}, \tilde{z}') \geq 3K$, $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$. We get the property P4.

1.4.4. Now we define the *linking number* $I(\tilde{F}; z, z') \in \mathbb{Z}$ for every distinct contractible fixed points z and z' of F as follows:

$$I(\tilde{F}; z, z') = \sum_{\alpha \in G} i(\tilde{F}; \tilde{z}, \alpha(\tilde{z}')),$$

where $\tilde{z} \in \pi^{-1}(z)$ and $\tilde{z}' \in \pi^{-1}(z')$. The sum is well defined since there are finite nonzero items in the sum (by the property P4). Obviously, $I(\tilde{F}; z, z')$ does not depend on the chosen lifts \tilde{z} and \tilde{z}' (by the property P2) and is locally constant on $(\text{Fix}_{\text{Cont},I}(F) \times \text{Fix}_{\text{Cont},I}(F)) \setminus \Delta$ (by the property P1 and the fact that there is a finite number of nonzero in the sum).

Proposition 1.5. *The following statements are equivalent*

- (1) *The set of linking numbers $i(\tilde{F}; \tilde{z}, \tilde{z}')$ where $(\tilde{z}, \tilde{z}') \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ is bounded;*
- (2) *The set of linking numbers $I(\tilde{F}; z, z')$ where $(z, z') \in (\text{Fix}_{\text{Cont},I}(F) \times \text{Fix}_{\text{Cont},I}(F)) \setminus \Delta$ is bounded.*

Proof. (1) \Rightarrow (2). Let N be a bound of that set. According to the property P4, there exists a positive integer K such that $\#\{\alpha \in G \mid i(\tilde{F}; \tilde{z}, \alpha(\tilde{z}')) \neq 0\} \leq K$ for all $\tilde{z}, \tilde{z}' \in \text{Fix}(\tilde{F})$. Then we have $|I(\tilde{F}; z, z')| \leq \sum_{\alpha \in G} |i(\tilde{F}; \tilde{z}, \alpha(\tilde{z}'))| \leq KN$.

(2) \Rightarrow (1). If the statement (1) does not hold, then there exist two sequences $\{\tilde{z}_n\}_{n \geq 1}$ and $\{\tilde{z}'_n\}_{n \geq 1}$ such that $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{z}_n, \tilde{z}'_n) = +\infty$ or $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{z}_n, \tilde{z}'_n) = -\infty$. We suppose $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{z}_n, \tilde{z}'_n) = +\infty$, the other case being similar. As M is compact, there is a subsequence $\{\tilde{z}_{n_k}\}_{k \geq 1}$ of $\{\tilde{z}_n\}_{n \geq 1}$ and a subsequence $\{\tilde{z}'_{n_k}\}_{k \geq 1}$ of $\{\tilde{z}'_n\}_{n \geq 1}$ such that $\pi(\tilde{z}_{n_k}) \rightarrow z$ and $\pi(\tilde{z}'_{n_k}) \rightarrow z'$ when $k \rightarrow +\infty$. By the continuity of I , we have $z, z' \in \text{Fix}_{\text{Cont},I}(F)$. Fix two points $\tilde{z} \in \pi^{-1}(z)$ and $\tilde{z}' \in \pi^{-1}(z')$. We can choose a sequence $\{\alpha_k\}_{k \geq 1} \subset G$ such that $\alpha_k(\tilde{z}_{n_k}) \rightarrow \tilde{z}$ as $k \rightarrow +\infty$. By the property P2, we have

$$(1.4.3) \quad \lim_{k \rightarrow +\infty} i(\tilde{F}; \alpha_k(\tilde{z}_{n_k}), \alpha_k(\tilde{z}'_{n_k})) = \lim_{k \rightarrow +\infty} i(\tilde{F}; \tilde{z}_{n_k}, \tilde{z}'_{n_k}) = +\infty.$$

The property P4 implies that the sequence $\{\alpha_k(\tilde{z}'_{n_k})\}_{k \geq 1}$ is bounded, then the property P1 tell us that $\lim_{k \rightarrow +\infty} \alpha_k(\tilde{z}'_{n_k}) = \tilde{z}$. By the properties P1 and P3, we have

$$i(\tilde{F}; \alpha_k(\tilde{z}_{n_k}), \alpha(\alpha_k(\tilde{z}'_{n_k}))) = 0$$

for all $\alpha \in G^*$ when k is large enough. Thus we have $\lim_{k \rightarrow +\infty} I(\tilde{F}; \pi(\tilde{z}_{n_k}), \pi(\tilde{z}'_{n_k})) = +\infty$. \square

1.4.5. In the rest of the paper, when we take two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , it does not mean that $\pi(\tilde{a})$ and $\pi(\tilde{b})$ are distinct.

Fix two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} . For any $z \in \text{Fix}_{\text{Cont}, I}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, we define the linking number of z for \tilde{a} and \tilde{b} as

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_{\pi(\tilde{z})=z} \left(i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}) \right) = I(\tilde{F}; \pi(\tilde{a}), z) - I(\tilde{F}; \pi(\tilde{b}), z).$$

We will extend it to the case where $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$ in Section 4. Note here that the linking number only depends on $\pi(\tilde{a})$ and $\pi(\tilde{b})$ in the case where z is a contractible fixed point of F , but the extension of $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for $z \in \text{Rec}^+(F) \setminus \text{Fix}_{\text{Cont}, I}(F)$ in Section 4 depends on the choices of \tilde{a} and \tilde{b} .

1.5. The weak boundedness property and the boundedness property. We can compactify \tilde{M} into a sphere by adding a point ∞ at infinity and the lift \tilde{F} may be extended by fixing this point. In all the text, we write $\mathbf{S} = \tilde{M} \sqcup \{\infty\}$. If \tilde{a} and \tilde{b} are distinct fixed points of \tilde{F} , the restriction of \tilde{F} to the annulus $A_{\tilde{a}, \tilde{b}} = \mathbf{S} \setminus \{\tilde{a}, \tilde{b}\}$ denoted by $\tilde{F}_{\tilde{a}, \tilde{b}}$, has a natural lift $\hat{F}_{\tilde{a}, \tilde{b}}$ to the universal cover $\hat{A}_{\tilde{a}, \tilde{b}}$ of $A_{\tilde{a}, \tilde{b}}$ that fixes the preimages of ∞ by the covering projection $\hat{\pi}_{\tilde{a}, \tilde{b}} : \hat{A}_{\tilde{a}, \tilde{b}} \rightarrow A_{\tilde{a}, \tilde{b}}$. Denote by $T_{\tilde{a}, \tilde{b}}$ the generator of $H_1(A_{\tilde{a}, \tilde{b}}, \mathbb{R})$ defined by the oriented boundary of a small disk centered at \tilde{a} .

If $\pi(\tilde{a}) \neq \pi(\tilde{b})$, by Remark 1.1, there exist two identity isotopies I' and I'' homotopic to I with fixed endpoints such that I' fixes $\pi(\tilde{a})$ and I'' fixes $\pi(\tilde{b})$. However, in general, there does not exist an identity isotopy I''' homotopic to I with fixed endpoints such that I''' fixes both $\pi(\tilde{a})$ and $\pi(\tilde{b})$, which is an obstacle that prevents us to generalize the action function to a more general cases (see Section 2.3). That is a reason that we introduce the following lemma.

Lemma 1.6. *If \tilde{z} is another fixed point of \tilde{F} which is different from \tilde{a} , \tilde{b} and ∞ , then the rotation number of $\tilde{z} \in A_{\tilde{a}, \tilde{b}}$ for the natural lift $\hat{F}_{\tilde{a}, \tilde{b}}$ is equal to $i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z})$, that is*

$$\rho_{A_{\tilde{a}, \tilde{b}}, \hat{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}) = i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}).$$

Proof. If J and J' are two isotopies of \tilde{M} from $\text{Id}_{\tilde{M}}$ to \tilde{F} , then there exists $k \in \mathbb{Z}$ such that $i_J = i_{J'} + k$ (see 1.4.2). Therefore, if \tilde{a}, \tilde{b} and \tilde{z} are distinct fixed points of \tilde{F} , the quantity $i_J(\tilde{a}, \tilde{z}) - i_J(\tilde{b}, \tilde{z})$ is independent of J and hence equals to $i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z})$ if we choose $J = \tilde{I}$ where \tilde{I} is the identity isotopy in 1.4.3. Suppose now that J is an isotopy that fixes \tilde{a} and \tilde{b} . The trajectory $J(\tilde{z})$ defines a loop in the sphere \mathbf{S} . If $\gamma_{\tilde{a}, \infty}$ and $\gamma_{\tilde{b}, \infty}$ are two paths in \mathbf{S} that join respectively \tilde{a} and \tilde{b} to ∞ , we have $i_J(\tilde{a}, \tilde{z}) = \gamma_{\tilde{a}, \infty} \wedge J(\tilde{z})$ and $i_J(\tilde{b}, \tilde{z}) = \gamma_{\tilde{b}, \infty} \wedge J(\tilde{z})$. The loop $J(\tilde{z})$ being homologous to zero in \mathbf{S} , we deduce that $i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}) = i_J(\tilde{a}, \tilde{z}) - i_J(\tilde{b}, \tilde{z}) = \gamma_{\tilde{a}, \tilde{b}} \wedge J(\tilde{z})$, where $\gamma_{\tilde{a}, \tilde{b}}$ is a path in \mathbf{S} that joins \tilde{a} to \tilde{b} . Note that this integer is nothing else but the rotation number of \tilde{z} for the lift $\hat{F}_{\tilde{a}, \tilde{b}}$ defined by $T_{\tilde{a}, \tilde{b}}$. \square

Remark here that, by the definition $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ of Section 1.4.5, we have

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_{\pi(\tilde{z})=z} i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}) = \sum_{\pi(\tilde{z})=z} \rho_{A_{\tilde{a}, \tilde{b}}, \hat{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}).$$

Definition 1.7. We say that I satisfies *the weak boundedness property at $\tilde{a} \in \text{Fix}(\tilde{F})$ (WB-property at \tilde{a})* if $i(\tilde{F}; \tilde{a}, \tilde{b})$ is uniformly bounded for all fixed point $\tilde{b} \in \text{Fix}(\tilde{F}) \setminus \{\tilde{a}\}$. We say that I satisfies *the weak boundedness property (WB-property)* if it satisfies the weak boundedness property at every $\tilde{a} \in \text{Fix}(\tilde{F})$. We say that I satisfies *the boundedness property (B-property)* if the set of $i(\tilde{F}; \tilde{a}, \tilde{b})$ where $(\tilde{a}, \tilde{b}) \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ is bounded.

Lemma 1.8. *Let \tilde{a} and \tilde{b} be two distinct fixed points of \tilde{F} . The following statements are equivalent*

- (1) *I satisfies the WB-property at \tilde{a} and \tilde{b} ;*
- (2) *there exists $K \geq 0$ such that $\left| \rho_{A_{\tilde{a}, \tilde{b}}, \hat{F}_{\tilde{a}, \tilde{b}}}(\tilde{c}) \right| \leq K$ for all fixed point $\tilde{c} \in \text{Fix}(\tilde{F}) \setminus \{\tilde{a}, \tilde{b}\}$.*

Proof. From Lemma 1.6, we have (1) \Rightarrow (2) immediately. Next we prove (2) \Rightarrow (1) by contradiction. Without loss of generality, we suppose that there exists a sequence $\{\tilde{c}_n\}_{n \geq 1} \subset \text{Fix}(\tilde{F}) \setminus \{\tilde{a}, \tilde{b}\}$ such that $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}, \tilde{c}_n) = +\infty$ (the case $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}, \tilde{c}_n) = -\infty$ is similar). Lemma 1.6 and the hypothesis (2) imply that $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{b}, \tilde{c}_n) = +\infty$. The property P4 implies that the sequence $\{\tilde{c}_n\}_{n \geq 1}$ is bounded. The property P1 implies that $\lim_{n \rightarrow +\infty} \tilde{c}_n = \tilde{a}$ and $\lim_{n \rightarrow +\infty} \tilde{c}_n = \tilde{b}$, which gives a contradiction. \square

Lemma 1.9. *For any two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , if F and F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$, then $\rho_{A_{\tilde{a}, \tilde{b}}, \hat{F}_{\tilde{a}, \tilde{b}}}(\tilde{z})$ is uniformly bounded for any $\tilde{z} \in \text{Rec}^+(\tilde{F}) \setminus \{\tilde{a}, \tilde{b}\}$ if it exists. In particular, $\rho_{A_{\tilde{a}, \tilde{b}}, \hat{F}_{\tilde{a}, \tilde{b}}}(\tilde{c})$ is uniformly bounded for any fixed point $\tilde{c} \in \text{Fix}(\tilde{F}) \setminus \{\tilde{a}, \tilde{b}\}$.*

Proof. Let $\bar{A}_{\tilde{a}, \tilde{b}} = S_{\tilde{a}} \sqcup A_{\tilde{a}, \tilde{b}} \sqcup S_{\tilde{b}}$ where $S_{\tilde{a}}$ and $S_{\tilde{b}}$ are the tangent unit circles at \tilde{a} and \tilde{b} such that $\bar{A}_{\tilde{a}, \tilde{b}}$ is the natural compactification of $A_{\tilde{a}, \tilde{b}}$. The maps F and F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$. Hence the lift \tilde{F} (resp. \tilde{F}^{-1}) of F (resp. F^{-1}) to \bar{M} is differentiable at \tilde{a} and \tilde{b} . By the method of *blowing-up*, it induces a homeomorphism $f : \bar{A}_{\tilde{a}, \tilde{b}} \rightarrow \bar{A}_{\tilde{a}, \tilde{b}}$,

$$f(u) = \begin{cases} \tilde{F}_{\tilde{a}, \tilde{b}}(u) & \text{when } u \in A_{\tilde{a}, \tilde{b}} \\ \frac{D\tilde{F}(\tilde{a}).u}{|D\tilde{F}(\tilde{a}).u|} & \text{when } u \in S_{\tilde{a}} \\ \frac{D\tilde{F}(\tilde{b}).u}{|D\tilde{F}(\tilde{b}).u|} & \text{when } u \in S_{\tilde{b}}. \end{cases}$$

The universal cover of $\bar{A}_{\tilde{a}, \tilde{b}}$ is $\mathbb{R} \times [0, 1]$. We suppose that \hat{f} is the lift of f fixing the preimages of ∞ by the covering projection $\hat{\pi}_{\tilde{a}, \tilde{b}}$. For any $u \in \bar{A}_{\tilde{a}, \tilde{b}}$, we have that $p_1(\hat{f}(\hat{u})) - p_1(\hat{u})$ is uniformly bounded because $\bar{A}_{\tilde{a}, \tilde{b}}$ is compact, where \hat{u} is any lift of u . There

exists N , depending on I , such that for every $\hat{z} \in \hat{A}_{\tilde{a}, \tilde{b}}$, one has $\left| p_1(\hat{F}_{\tilde{a}, \tilde{b}}(\hat{z})) - p_1(\hat{z}) \right| \leq N$. Moreover, for every $n \geq 1$, we have

$$\left| \frac{p_1 \circ \hat{F}_{\tilde{a}, \tilde{b}}^n(\hat{z}) - p_1(\hat{z})}{n} \right| \leq \frac{1}{n} \sum_{i=0}^{n-1} \left| p_1 \circ \hat{F}_{\tilde{a}, \tilde{b}}^{i+1}(\hat{z}) - p_1 \circ \hat{F}_{\tilde{a}, \tilde{b}}^i(\hat{z}) \right| \leq N.$$

If $\tilde{z} \in \text{Rec}^+(\tilde{F}_{\tilde{a}, \tilde{b}})$ and $\rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\tilde{z})$ exists, by the definition of rotation number (see 1.3.3), we deduce that $\left| \rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}) \right| \leq N$. We have completed the proof. \square

Observe that the proof of Lemma 1.9 gives us an information about how rotate not only the positively recurrent points of $\tilde{F}_{\tilde{a}, \tilde{b}}$ but in fact every point in $A_{\tilde{a}, \tilde{b}}$, we will use this fact in Section 5.

By Lemma 1.8 and Lemma 1.9, we have the following proposition immediately.

Proposition 1.10. *The WB-property is satisfied if $F \in \text{Diff}(M)$.*

Remark that if F and F^{-1} are differentiable at $\pi(\tilde{a})$, similarly to the proof of Lemma 1.9, we can prove that I satisfies the WB-property at \tilde{a} , from which Proposition 1.10 can be proven directly. However, the proof of the current Lemma 1.9 will be necessary for further proofs of this paper, thus being adopted here.

Obviously, I satisfies the B-property if $\sharp \text{Fix}_{\text{Cont}, I}(F) < +\infty$. In Appendix, we construct an isotopy $I = (F_t)_{0 \leq t \leq 1}$ such that $F = F_1$ is a diffeomorphism of M but does not satisfy the B-property. In that example, we show that F is not a C^1 -diffeomorphism of M (see Example 8.3). If F is a C^1 -diffeomorphism of M , we have the following result:

Proposition 1.11. *The B-property is satisfied if $F \in \text{Diff}^1(M)$.*

Before proving Proposition 1.11, we need the following lemma ([BLFM, Lemma 5.6]).

Lemma 1.12. *Let h be a C^1 -diffeomorphism of \mathbf{S}^2 and $a \in \text{Fix}(h)$. For all point $z \in \mathbf{S}^2$ different from a and its antipodal point, denote γ_z the unique great circle that passes through them and a , and denote γ_z^- (resp. γ_z^+) the small (resp. large) arc of γ_z joining a and z . Then there exists a neighborhood W of a on \mathbf{S}^2 such that for all $z \in \text{Fix}(h) \cap W$, we have $h(\gamma_z^-) \cap \gamma_z^+ = \{z, a\}$.*

Proof of Proposition 1.11. We only need to consider the case where $\sharp \text{Fix}_{\text{Cont}, I}(F) = +\infty$. To get a proof by contradiction, according to Definition 1.7, we suppose that there exist a sequence of pairs $\{(\tilde{a}_n, \tilde{b}_n)\}_{n \geq 1} \subset (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ such that $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{b}_n) = +\infty$ (the case where $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{b}_n) = -\infty$ is similar). By the property P2, we can suppose that the sequence $\{\tilde{a}_n\}_{n \geq 1}$ is bounded by replacing \tilde{a}_n and \tilde{b}_n with $\alpha_n(\tilde{a}_n)$ and $\alpha_n(\tilde{b}_n)$ where $\alpha_n \in G$ if necessary. The property P4 implies that the sequence $\{\tilde{b}_n\}_{n \geq 1}$ is also bounded. Therefore, by continuity, we can suppose that $\lim_{n \rightarrow +\infty} \tilde{a}_n = \tilde{a}$ and $\lim_{n \rightarrow +\infty} \tilde{b}_n = \tilde{b}$ where $\tilde{a} \in \text{Fix}(\tilde{F})$ and $\tilde{b} \in \text{Fix}(\tilde{F})$ by extracting subsequences if necessary. According to

the property P1, we deduce that $\tilde{a} = \tilde{b}$. Moreover, as F is a diffeomorphism, so I satisfies the WB-property at \tilde{a} . That is, there is a number $N_{\tilde{a}} \geq 0$ such that $|i(\tilde{F}; \tilde{a}, \tilde{z})| \leq N_{\tilde{a}}$ for all $\tilde{z} \in \text{Fix}(\tilde{F}) \setminus \{\tilde{a}\}$. Hence, we can suppose that $\tilde{a}_n \neq \tilde{a}$ and $\tilde{b}_n \neq \tilde{a}$ for all n by taking n large enough.

For every $n \geq 1$, let \tilde{I}_n be an isotopy that fixes \tilde{a} and \tilde{a}_n (Corollary 1.3). Then there exists k_n such that

$$(1.5.1) \quad i_{\tilde{I}_n}(\tilde{z}, \tilde{z}') = i(\tilde{F}; \tilde{z}, \tilde{z}') + k_n$$

for every two distinct fixed points \tilde{z} and \tilde{z}' of \tilde{F} (see 1.4.2). Observing that $i_{\tilde{I}_n}(\tilde{a}, \tilde{a}_n) = 0$ for every n , so Equation 1.5.1 implies that $|k_n| \leq N_{\tilde{a}}$ and $\lim_{n \rightarrow +\infty} i_{\tilde{I}_n}(\tilde{a}_n, \tilde{b}_n) = +\infty$. Moreover, we have $i_{\tilde{I}_n}(\tilde{a}, \tilde{b}_n) = i(\tilde{F}; \tilde{a}, \tilde{b}_n) + k_n$, hence $|i_{\tilde{I}_n}(\tilde{a}, \tilde{b}_n)| \leq 2N_{\tilde{a}}$.

Consider the annulus $A_{\tilde{a}, \tilde{a}_n} = \mathbf{S} \setminus \{\tilde{a}, \tilde{a}_n\}$ and $\tilde{F}_{\tilde{a}, \tilde{a}_n}$. By the proof of Lemma 1.6, we know that

$$\rho_{A_{\tilde{a}, \tilde{a}_n}, \hat{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{b}_n) = i_{\tilde{I}_n}(\tilde{a}, \tilde{b}_n) - i_{\tilde{I}_n}(\tilde{a}_n, \tilde{b}_n).$$

Therefore,

$$(1.5.2) \quad \lim_{n \rightarrow +\infty} \rho_{A_{\tilde{a}, \tilde{a}_n}, \hat{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{b}_n) = -\infty.$$

Fix $q \geq 1$, apply Lemma 1.12 to $\tilde{F}_{\tilde{a}, \tilde{a}_n}$. When n is large enough, there are two arcs $\tilde{\gamma}^-$ and $\tilde{\gamma}^+$ in $A_{\tilde{a}, \tilde{a}_n}$ joining \tilde{a} and \tilde{a}_n that are disjoint and $\tilde{F}_{\tilde{a}, \tilde{a}_n}^q(\tilde{\gamma}^-) \cap \tilde{\gamma}^+ = \emptyset$. Recall that $\hat{\pi}_{\tilde{a}, \tilde{a}_n} : \hat{A}_{\tilde{a}, \tilde{a}_n} \rightarrow A_{\tilde{a}, \tilde{a}_n}$ is the universal cover of $A_{\tilde{a}, \tilde{a}_n}$, $\hat{F}_{\tilde{a}, \tilde{a}_n}$ is the lift of $\tilde{F}_{\tilde{a}, \tilde{a}_n}$ that fixes the preimages of ∞ by $\hat{\pi}_{\tilde{a}, \tilde{a}_n}$ and $T_{\tilde{a}, \tilde{a}_n}$ is the generator of $H_1(A_{\tilde{a}, \tilde{a}_n}, \mathbb{R})$ defined by the oriented boundary of small disk centered at \tilde{a} . Choose a connected component $\hat{\gamma}^-$ of $\hat{\pi}_{\tilde{a}, \tilde{a}_n}^{-1}(\tilde{\gamma}^-)$ and endow $\hat{\gamma}^-$ with an orientation from the lower end to the upper end. The arc $\hat{F}_{\tilde{a}, \tilde{a}_n}^q(\hat{\gamma}^-)$ does not meet any connected component of $\hat{\pi}_{\tilde{a}, \tilde{a}_n}^{-1}(\tilde{\gamma}^+)$ and thus meets at most a translated $T_{\tilde{a}, \tilde{a}_n}^k(\hat{\gamma}^-)$. As $\hat{F}_{\tilde{a}, \tilde{a}_n}$ has a fixed point (the lift $\hat{\infty}$ of ∞), the arc $\hat{F}_{\tilde{a}, \tilde{a}_n}^q(\hat{\gamma}^-)$ can not be on the right of $T_{\tilde{a}, \tilde{a}_n}^q(\hat{\gamma}^-)$ (otherwise, $\hat{F}_{\tilde{a}, \tilde{a}_n}^q$ has no fixed point). Therefore, it is on the left of the arc $T_{\tilde{a}, \tilde{a}_n}^2(\hat{\gamma}^-)$. For the same reason, it is on the right of the arc $T_{\tilde{a}, \tilde{a}_n}^{-2}(\hat{\gamma}^-)$. As $\hat{F}_{\tilde{a}, \tilde{a}_n}$ and $T_{\tilde{a}, \tilde{a}_n}$ commute, it implies that the arc $\hat{F}_{\tilde{a}, \tilde{a}_n}^q(T(\hat{\gamma}^-))$ is on the left of $T_{\tilde{a}, \tilde{a}_n}^3(\hat{\gamma}^-)$ and on the right of $T_{\tilde{a}, \tilde{a}_n}^{-1}(\hat{\gamma}^-)$. Consider a point $\tilde{z} \in \text{Rec}^+(\tilde{F}) \setminus \{\tilde{a}, \tilde{a}_n\}$ such that the rotation number $\rho_{A_{\tilde{a}, \tilde{a}_n}, \hat{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{z})$ is well defined. There exists a unique lift \hat{z} of \tilde{z} that is in the region between $\hat{\gamma}^-$ and $T_{\tilde{a}, \tilde{a}_n}(\hat{\gamma}^-)$. By induction, we deduce that the point $\hat{F}_{\tilde{a}, \tilde{a}_n}^{qm}(\hat{z})$ is in the region between $T_{\tilde{a}, \tilde{a}_n}^{-2m}(\hat{\gamma}^-)$ and $T_{\tilde{a}, \tilde{a}_n}^{3m}(\hat{\gamma}^-)$ for all $m \geq 1$. By the definition of the rotation number (see 1.3.3), we have $|\rho_{A_{\tilde{a}, \tilde{a}_n}, \hat{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{z})| \leq 3/q$. As q can be choose arbitrarily large, we have

$$(1.5.3) \quad \lim_{n \rightarrow +\infty} \rho_{A_{\tilde{a}, \tilde{a}_n}, \hat{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{z}) = 0.$$

In particular, we have

$$\lim_{n \rightarrow +\infty} \rho_{A_{\tilde{a}, \tilde{a}_n}, \hat{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{b}_n) = 0,$$

which conflicts with the limit 1.5.2. We have completed the proof. \square

2. SYMPLECTIC ACTION

The action is a classical object in symplectic geometry and we will first recall it in this section. Then, we will explain how to generalize the action to a simple case where the time-one map F of I is a diffeomorphism, the set $\text{Fix}_{\text{Cont},I}(F)$ of contractible fixed points is finite and unlinked (we will define what it means), the measure $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont},I}(F)$ and satisfies $\rho_{M,I}(\mu) = 0$.

Suppose that I is an identity isotopy of M , F is the time-one map of I , $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont},I}(F)$, and $\rho_{M,I}(\mu) = 0$. At the end of the section, we will state generalizations of the action to the cases

- $F \in \text{Diff}(M)$;
- I satisfies the WB-property, the measure μ has total support;
- I satisfies the WB-property, the measure μ is ergodic.

We will prove them in Section 6.

2.1. The classical action. Let us recall what is the action. In this section, we suppose that (M, ω) is a symplectic manifold (not necessarily closed).

2.1.1. Symplectic and Hamiltonian. A diffeomorphism $F : M \rightarrow M$ is called *symplectic* if it preserves the form ω . Symplectic diffeomorphisms form a group denoted by $\text{Symp}(M, \omega)$. Let $\text{Symp}_*(M, \omega)$ denote the path-connected component of the Id_M in $\text{Symp}(M, \omega)$.

Consider a smooth isotopy $I = (F_t)_{t \in [0,1]}$ in $\text{Symp}_*(M, \omega)$ with $F_0 = \text{Id}_M$ and $F_1 = F$. Let ξ_t be the corresponding time-dependent vector field on M :

$$\frac{d}{dt}F_t(x) = \xi_t(F_t(x)) \quad \text{for all } x \in M, \quad t \in [0, 1].$$

Since the Lie derivative $L_{\xi_t}\omega$ vanishes, we get that the 1-forms $\lambda_t = -i_{\xi_t}\omega$ are closed. Write $[\lambda_t]$ for the cohomology class of λ_t . The quantity

$$\text{Flux}(I) = \int_0^1 [\lambda_t] dt \in H^1(M, \mathbb{R}),$$

is called the *flux* of the isotopy I . It is well known that $\text{Flux}(I)$ does not change under a homotopy of the path I with fixed end points (see [MS]).

An isotopy I is called *Hamiltonian* if the 1-forms λ_t are exact for all t . In this case there exists a smooth function $H : [0, 1] \times M \rightarrow \mathbb{R}$ so that $\lambda_t = dH_t$, where $H_t(x)$ stands for $H(t, x)$. The function H is called the Hamiltonian function generating the flow I . Note that H_t is defined uniquely up to an additive time-dependent constant.

A symplectic diffeomorphism $F : M \rightarrow M$ is called *Hamiltonian* if there exists a Hamiltonian isotopy $I = (F_t)_{t \in [0,1]}$ with $F_0 = \text{Id}_M$ and $F_1 = F$. Hamiltonian diffeomorphisms form a group denoted by $\text{Ham}(M, \omega)$. The following theorem characterizes the relation between flux and Hamiltonian diffeomorphisms (see [MS] for the details).

Theorem 2.1. *Let $F \in \text{Symp}_*(M, \omega)$. Then F is Hamiltonian if and only if there exists an isotopy $I = (F_t)_{t \in [0,1]}$ in $\text{Symp}_*(M, \omega)$ such that $F_0 = \text{Id}_M$, $F_1 = F$ and $\text{Flux}(I) = 0$. In that case, I is isotopic with fixed endpoints to a Hamiltonian isotopy.*

Suppose that (M, ω) is a closed symplectic surface and $I = (F_t)_{t \in [0,1]}$ is a smooth isotopy in $\text{Symp}_*(M, \omega)$. Let denote by μ the measure induced by ω . We have the following relation between the $\text{Flux}(I)$ and $\rho_{M,I}(\mu)$ (see [FH1]): for any smooth loop σ on M , we have

$$\langle \text{Flux}(I), [\sigma]_M \rangle = \rho_{M,I}(\mu) \wedge [\sigma]_M.$$

Hence, I is Hamiltonian if and only if $\rho_{M,I}(\mu) = 0$.

2.1.2. Action function and action difference. In this section, we suppose that (M, ω) is a symplectic manifold with $\pi_2(M) = 0$ (for example, a closed oriented surface with genus $g \geq 1$).

Let $I = (F_t)_{t \in [0,1]}$ be a Hamiltonian isotopy on M with $F_0 = \text{Id}_M$ and $F_1 = F$. Suppose that the function H is the Hamiltonian function generating the flow I .

Let x be a contractible fixed point of F . Take any immersed disk $D_x \subset M$ with $\partial D_x = I(x)$, and define the *action function*

$$(2.1.1) \quad \mathcal{A}_H(x) = \int_{D_x} \omega - \int_0^1 H_t(F_t(x)) dt.$$

The definition is well defined, that is $\mathcal{A}_H(x)$ does not depend on the choice of D_x . It is sufficient to prove the integral $\int_{D_x} \omega$ does not depend on the choice of D_x . Indeed, let D'_x be another choice, the 2-chain $\Pi = D_x - D'_x$ represents an immersed 2-sphere in M , and hence $\int_{\Pi} \omega = 0$ since $\pi_2(M) = 0$. Hence the claim follows.

Given two contractible fixed points x and y of F , take a path $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = x$ and $\gamma(1) = y$. Choose two immersed disks D_x and D_y so that $\partial D_x = I(x)$ and $\partial D_y = I(y)$. Let us define $\Delta : [0, 1] \times [0, 1] \rightarrow M$ by $\Delta(t, s) = F_t(\gamma(s))$ where we assume that the boundary of the square $[0, 1] \times [0, 1]$ is oriented counter-clockwise and observe that $\partial \Delta = -\gamma + F\gamma - I(y) + I(x)$. So $F\gamma - \gamma = \partial \Delta + \partial D_y - \partial D_x$ is a 1-cycle and is the boundary of Σ where Σ is a 2-chain.

Define the *action difference* for x and y :

$$(2.1.2) \quad \delta(F; x, y) = \int_{\Sigma} \omega.$$

Since $\pi_2(M) = 0$, it does not depend on the choice of Σ , and hence not on D_x and D_y . Let us prove that it does not depend on the choice of γ .

Denote by ξ_t the vector field of the flow F_t (see 2.1.1). Then

$$\begin{aligned} \Delta^* \omega &= \omega \left(\xi_t(F_t \gamma(s)), \frac{\partial}{\partial s} F_t \gamma(s) \right) dt \wedge ds \\ &= -dH_t \left(\frac{\partial}{\partial s} F_t \gamma(s) \right) dt \wedge ds. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{\Delta} \omega &= \int_{[0,1]^2} \Delta^* \omega = - \int_0^1 dt \int_0^1 dH_t \left(\frac{\partial}{\partial s} F_t \gamma(s) \right) ds \\ &= \int_0^1 H_t(F_t(x)) dt - \int_0^1 H_t(F_t(y)) dt. \end{aligned}$$

Finally, we have

$$(2.1.3) \quad \delta(F; x, y) = \int_{\Sigma} \omega = \int_{\Delta} \omega + \int_{D_y} \omega - \int_{D_x} \omega = \mathcal{A}_H(y) - \mathcal{A}_H(x).$$

Equation 2.1.3 shows that the action difference does not depend on the choice of γ , we have completed our claim. Moreover, we also give a relation between the action difference and the action function.

2.1.3. The action function and action difference on the universal covering space. When $I = (F_t)_{t \in [0,1]} \subset \text{Symp}_*(M, \omega) \setminus \text{Ham}(M, \omega)$, the action function (see Definition 2.1.1) is not meaningful. However, observing that the universal cover \widetilde{M} of M is simply connected, the lifted identity isotopy $\widetilde{I} = (\widetilde{F}_t)_{t \in [0,1]} \subset \text{Symp}_*(\widetilde{M}, \widetilde{\omega})$ of I to \widetilde{M} where $\widetilde{\omega}$ is the lift of the symplectic structure ω to \widetilde{M} is automatically Hamiltonian since $H^1(\widetilde{M}, \mathbb{R}) = 0$ (see Theorem 2.1). Let \widetilde{H} be the Hamiltonian function generating the flow \widetilde{I} . As before, we can define the action function $\mathcal{A}_{\widetilde{H}}(\widetilde{x})$ for any fixed point \widetilde{x} of $\widetilde{F} = \widetilde{F}_1$ and the action difference $\delta(\widetilde{F}; \widetilde{x}, \widetilde{y})$ for any two distinct fixed points \widetilde{x} and \widetilde{y} of \widetilde{F} , and we have the relation $\delta(\widetilde{F}; \widetilde{x}, \widetilde{y}) = \mathcal{A}_{\widetilde{H}}(\widetilde{y}) - \mathcal{A}_{\widetilde{H}}(\widetilde{x})$.

Let us see what happens in the particular case where I is Hamiltonian. Suppose that H is the Hamiltonian function generating the flow I and \widetilde{H} is its lift to \widetilde{M} . For any contractible fixed point x of F and its lift \widetilde{x} , we have $\mathcal{A}_{\widetilde{H}}(\widetilde{x}) = \mathcal{A}_H(x)$ (see [P1, Theorem 2.1.C] and [FH1, Remark 2.7]). Hence, for any two distinct contractible fixed points x and y of F , and their lifts \widetilde{x} and \widetilde{y} , we have

$$(2.1.4) \quad \delta(\widetilde{F}; \widetilde{x}, \widetilde{y}) = \mathcal{A}_{\widetilde{H}}(\widetilde{y}) - \mathcal{A}_{\widetilde{H}}(\widetilde{x}) = \mathcal{A}_H(y) - \mathcal{A}_H(x).$$

2.2. A generalization of the action function. The action difference of two contractible fixed points x, y of F equals to the algebraic area of any path γ connecting x and y along the isotopy I , that is, the area of the path γ along I swept out. By this observation, we would like to generalize such an object to the case where ω is replaced by a finite Borel measure μ and the Hamiltonian isotopy by an identity isotopy I with $\rho_{M,I}(\mu) = 0$.

There is a case where this can be done easily (see [Lec1]). Suppose that $I = (F_t)_{t \in [0,1]}$ is an identity isotopy of M , the time-one map F of I is a diffeomorphism, the set $\text{Fix}_{\text{Cont},I}(F)$ of contractible fixed points is finite and unlinked, that means that there exists an isotopy $I' = (F'_t)_{t \in [0,1]}$ homotopic to I that fixes every point of $\text{Fix}_{\text{Cont},I}(F)$, the measure $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont},I}(F)$ and satisfies $\rho_{M,I}(\mu) = 0$.

Let $N = M \setminus \text{Fix}_{\text{Cont},I}(F)$, by the method of blowing-up, we can naturally get a compactification \overline{N} of N if we replace each point $x \in \text{Fix}_{\text{Cont},I}(F)$ by S_x , the tangent unit

circle at x . The diffeomorphism $F|_N$ can be extended to a homeomorphism \overline{F} on \overline{N} which is isotopic to identity and induces the natural action by the linear map $DF(x)$ on S_x . As μ does not charge any point of $\text{Fix}_{\text{Cont},I}(F)$, we can define a measure on \overline{N} which is invariant by \overline{F} , denoted also μ . Therefore, we can define the rotation vector in $H_1(\overline{N}, \mathbb{R})$. The inclusion $\iota : N \hookrightarrow \overline{N}$ induces an isomorphism $\iota_* : H_1(N, \mathbb{R}) \rightarrow H_1(\overline{N}, \mathbb{R})$. We denote by $\rho_{N,I}(\mu) \in H_1(N, \mathbb{R})$ the rotation vector transported by this isomorphism. Let γ be a simple path in N joining $a \in \text{Fix}_{\text{Cont},I}(F)$ and $b \in \text{Fix}_{\text{Cont},I}(F)$. We can define the algebraic intersection number $\gamma \wedge \rho_{N,I}(\mu)$. Remark here that $\gamma \wedge \rho_{N,I}(\mu)$ is independent on the chosen γ because the rotation vector $\rho_{M,I}(\mu) \in H_1(M, \mathbb{R})$ is zero. Moreover, we can write

$$\gamma \wedge \rho_{N,I}(\mu) = L(b) - L(a),$$

where $L : \text{Fix}_{\text{Cont},I}(F) \rightarrow \mathbb{R}$ is a function, defined up to an additive constant. We call that L is the *action function*.

2.3. Our main theorem. It is natural to ask if we can generalize the action to a more general case. Let us first analyze what has been done above. The key points of his generalization are that F is a diffeomorphism of M and that there is another identity isotopy I' homotopic to I that fixes all contractible fixed points of F . The differentiability hypothesis prevents the dynamics to be too wild in a neighborhood of a contractible fixed point so that it provides some boundedness condition, which means one can compactify the sub-manifold $N = M \setminus \text{Fix}_{\text{Cont},I}(F)$ by blowing-up. It seems to us that keeping the boundedness condition is necessary and that is why we define the boundedness properties in 1.5. However, in general case, there may not exist such an isotopy I' that fixes all contractible fixed points of F . How to deal with this obstacle? The section 2.1.3 reminds us that it will be a good idea if we consider the universal covering space \widetilde{M} . A key point is that we can always find an isotopy \widetilde{I}' from $\text{Id}_{\widetilde{M}}$ to \widetilde{F} that fixes any two fixed points of \widetilde{F} , where \widetilde{F} is the time-one map of the lifted identity isotopy \widetilde{I} of I to \widetilde{M} (Corollary 1.3). It makes us able to define the action difference for every two fixed points of \widetilde{F} and generalize the classical action. Our main result is following.

Theorem 0.1 *Let M be a closed oriented surface with genus $g \geq 1$ and F be the time-one map of an identity isotopy I on M . Suppose that μ is a Borel finite measures on M that is invariant by F , has no atoms on $\text{Fix}_{\text{Cont},I}(F)$ and $\rho_{M,I}(\mu) = 0$. In all of the following cases*

- $F \in \text{Diff}(M)$;
- I satisfies the WB-property, the measure μ has total support;
- I satisfies the WB-property, the measure μ is ergodic,

an action function can be defined which generalizes the classical case.

We will prove it in Section 6.

3. DISK CHAINS

In this section, we will recall some classical results of the plane and the open annulus, and extend some results of Franks so that we can use them in Section 5.

Let M be a surface and let h be a homeomorphism of M . A *disk chain* C of h in M is given by a family $\{D_i\}_{1 \leq i \leq n}$ of embedded open disks of M and a family $\{m_i\}_{1 \leq i \leq n}$ of positive integers satisfying

- (1) if $i \neq j$, then either $D_i = D_j$ or $D_i \cap D_j = \emptyset$;
- (2) for $1 \leq i < n$, $h^{m_i}(D_i) \cap D_{i+1} \neq \emptyset$.

We will write $C = \{D_i\}_{1 \leq i \leq n}$ or $C = (\{D_i\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i \leq n})$ in a more detailed way. If $D_1 = D_n$ we will say that $\{D_i\}_{1 \leq i \leq n}$ is a *periodic disk chain*. We define the *length* of the chain C to be the integer $l(C) = \sum_{i=1}^{n-1} m_i$.

A *free disk* of h is a disk in M which does not meet its image by h . A *free disk chain* of h is a disk chain $C = \{D_i\}_{1 \leq i \leq n}$ such that every D_i is a free disk of h .

Recall the following fundamental result:

Proposition 3.1 (Franks' Lemma [Fr1]). *Let $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an orientation preserving homeomorphism. If H possesses a periodic free disk chain, then H has at least one fixed point.*

Recall that $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ is the open annulus and $T : (x, y) \mapsto (x+1, y)$ is the generator of the covering transformation group. Let $h \in \text{Homeo}_*(\mathbb{A})$ and H be a lift of h to \mathbb{R}^2 . We say that $\tilde{D} \subset \mathbb{R}^2$ is a *positively returning disk* if all the following conditions hold:

- $T^k(\tilde{D}) \cap \tilde{D} = \emptyset$ for all $k \in \mathbb{Z} \setminus \{0\}$;
- $H(\tilde{D}) \cap \tilde{D} = \emptyset$;
- there exist $n > 0$ and $k > 0$ such that $H^n(\tilde{D}) \cap T^k(\tilde{D}) \neq \emptyset$.

A *negatively returning disk* is defined similarly but with $k < 0$.

If there exists an open disk that is both positively and negatively returning, then it is easy to construct a periodic free disk chain of H . Hence, by Franks' Lemma, we have the following result (see [Fr1] for the detail):

Corollary 3.2. *If H has an open disk $\tilde{D} \subset \mathbb{R}^2$ which is both positively and negatively returning, then there is a fixed point of H .*

Suppose that $D \subset \mathbb{A}$ is a free disk of h , we define the following set:

$$(3.0.1) \quad \text{Rot}_D(H) = \text{Conv}\{p/q \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \setminus \{0\}, H^q(\tilde{D}) \cap T^p(\tilde{D}) \neq \emptyset\}$$

where $\text{Conv}(A)$ represents the convex hull of the set A and \tilde{D} is an arbitrary connected component of $\pi^{-1}(D)$. Observe here that $\text{Rot}_D(H)$ does not depend on the choice of \tilde{D} . By Corollary 3.2, we have the following result:

Corollary 3.3. *For every $k \in \text{Rot}_D(H) \cap \mathbb{Z}$, there exists a point \tilde{z}_0 such that $H(\tilde{z}_0) = T^k(\tilde{z}_0)$.*

Proof. Choose any connected component \tilde{D} of $\pi^{-1}(D)$. We first suppose that there is an integer k such that $H^q(\tilde{D}) \cap T^{kq}(\tilde{D}) \neq \emptyset$. Note here that this case covers the case where

k is a boundary point of $\text{Rot}_D(H)$. Denote by H' the lift $H' = T^{-k} \circ H$ of h . We have $H'^q(\tilde{D}) \cap \tilde{D} \neq \emptyset$ and $H'(\tilde{D}) \cap \tilde{D} = \emptyset$ since D is free. According to Proposition 3.1, H' has a fixed point \tilde{z}_0 , that is, $H(\tilde{z}_0) = T^k(\tilde{z}_0)$.

We now suppose that there are two rational numbers p_i/q_i ($i = 1, 2$) and an integer k such that

- $p_1/q_1 < k < p_2/q_2$;
- $H^{q_1}(\tilde{D}) \cap T^{p_1}(\tilde{D}) \neq \emptyset$;
- $H^{q_2}(\tilde{D}) \cap T^{p_2}(\tilde{D}) \neq \emptyset$.

Considering the lift $H' = T^{-k} \circ H$, we have

$$H'^{q_1}(\tilde{D}) \cap T^{p_1 - q_1 k}(\tilde{D}) \neq \emptyset$$

and

$$H'^{q_2}(\tilde{D}) \cap T^{p_2 - q_2 k}(\tilde{D}) \neq \emptyset.$$

Therefore, \tilde{D} is a both positively and negatively returning disk of H' . By Corollary 3.2, H' has a fixed point. We have completed the proof. \square

Let $C = (\{D_i\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i < n})$ be a periodic disk chain of h in \mathbb{A} . A lift of C for H in \mathbb{R}^2 is a disk chain $\tilde{C} = (\{\tilde{D}_i\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i < n})$ in \mathbb{R}^2 such that $\pi(\tilde{D}_i) = D_i$ for every i .

We define the *width* of the lift \tilde{C} of C to be the integer $w(H; \tilde{C}) = k$ such that $\tilde{D}_n = T^k(\tilde{D}_1)$. For every $p \in \mathbb{Z}$, the disk chain $T^p(\tilde{C}) = (\{T^p(\tilde{D}_i)\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i < n})$ is also a lift of C for H since H commutes with T . The disk chain

$$T^p \cdot \tilde{C} = \{\tilde{D}_1, T^{pm_1}(\tilde{D}_2), T^{p(m_1+m_2)}(\tilde{D}_3), \dots, T^{pl(C)}(\tilde{D}_n)\}$$

is a lift of C for $T^p \circ H$. Therefore, the width of \tilde{C} satisfies

$$w(H; \tilde{C}) = w(H; T^p(\tilde{C}))$$

and

$$w(T^p \circ H; T^p \cdot \tilde{C}) = p l(C) + w(H; \tilde{C})$$

for every $p \in \mathbb{Z}$.

Using Corollary 3.2 and Corollary 3.3, we have the following lemma.

Lemma 3.4. *Let $h \in \text{Homeo}_*(\mathbb{A})$ and H be a lift of h to \mathbb{R}^2 . Suppose that $\text{Rot}_{\text{Fix}(h)}(H) \subset [-N, N]$ for some $N \in \mathbb{N}$, and that there is a disk D in \mathbb{A} satisfying $H(\tilde{D}) \cap T^k(\tilde{D}) \neq \emptyset$ if and only if $k = 0$, where \tilde{D} is any connected component of $\pi^{-1}(D)$, and that a periodic disk chain $C = (\{D_i\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i < n})$ of h such that*

- (1) $D_1 = D$;
- (2) if $D_i \neq D$ then D_i is a free disk of h .

Then, we have

- $|w(H; \tilde{C})| < (N+1)l(C)$ for all lift \tilde{C} of C ;
- $\text{Rot}_{D_i}(H) \subset]-(N+1), N+1[$ if $D_i \neq D$.

Proof. Obviously, $C' = (\{D, D\}, \{1\})$ is a periodic disk chain of h .

Fix a connected component \tilde{D} of $\pi^{-1}(D)$ and a lift $\tilde{C} = \{\tilde{D}_i\}_{1 \leq i \leq n}$ of C for H that satisfies $\tilde{D}_1 = \tilde{D}$. Define \mathcal{D} as the family of all connected components of $\pi^{-1}(D_i)$, $1 \leq i \leq n$.

Suppose first that $w(H; \tilde{C}) \geq 0$, consider the lift $H' = H \circ T^{-(N+1)}$, we have the following facts

- $\text{Fix}(H') = \emptyset$;
- $H'(\tilde{D}) \cap \tilde{D} = \emptyset$;
- there is a free disk chain \tilde{C}' in \mathcal{D} of length 1 from \tilde{D} to $T^{-(N+1)}(\tilde{D})$ for H' (indeed, this disk chain is a lift of C' for H');
- there is a free disk chain \tilde{C} in \mathcal{D} of length $l(C)$ from \tilde{D} to $T^{-(N+1)l(C)+w(H; \tilde{C})}(\tilde{D})$ for H' (indeed, this disk chain is a lift of C for H').

The first item follows from $\text{Rot}_{\text{Fix}(h)}(H) \subset [-N, N]$. The second and third items hold by the hypothesis of D . The last one follows from the hypothesis (1) and the property of $w(H; \tilde{C})$.

If $-(N+1)l(C) + w(H; \tilde{C}) = 0$, then \tilde{C} is a periodic free disk chain for H' . By Proposition 3.1, H' has a fixed point, which conflicts with the first item. If $r = -(N+1)l(C) + w(H; \tilde{C}) > 0$, then the disk chain

$$\tilde{C} \cup T^r(\tilde{C}) \cup \dots \cup T^{Nr}(\tilde{C}) \cup T^{(N+1)r}(\tilde{C}') \cup \dots \cup T^{N+1}(\tilde{C}')$$

is a periodic free disk chain for H' . By Proposition 3.1 again, H' has a fixed point, which still conflicts with the first item. Hence $w(H; \tilde{C}) < (N+1)l(C)$.

In the case where $w(H; \tilde{C}) < 0$, replacing $H' = H \circ T^{-(N+1)}$ by $H' = H \circ T^{N+1}$, similarly to the case $w(H; \tilde{C}) \geq 0$, we get $w(H; \tilde{C}) > -(N+1)l(C)$. The first conclusion is proven.

Fix a disk $D_i \neq D$ and $p/q \in \text{Rot}_{D_i}(H)$. For every $s \geq 1$, consider the following periodic disk chain of h

$$C_s = \{D_1, \dots, \underbrace{D_i, \dots, D_i}_{s+1}, \dots, D_n\}$$

with

$$\{m_1, \dots, m_{i-1}, \underbrace{q, \dots, q}_s, m_i, \dots, m_{n-1}\}$$

and its lift for H

$$\tilde{C}_s = \{\tilde{D}_1, \dots, \tilde{D}_i, T^p(\tilde{D}_i), \dots, T^{sp}(\tilde{D}_i), T^{sp}(\tilde{D}_{i+1}), \dots, T^{sp}(\tilde{D}_n)\}.$$

Then we have $l(C_s) = l(C) + sq$ and $w(H; \tilde{C}_s) = w(H; \tilde{C}) + sp$. By the first conclusion, we get $|w(H; \tilde{C}_s)| < (N+1)l(C_s)$. Letting s tend to $+\infty$, we get $|p/q| \leq N+1$. Moreover, if $p/q = N+1$ (resp. $p/q = -(N+1)$), according to Corollary 3.3, then there exists a fixed point of h with rotation number $N+1$ (resp. $-(N+1)$) for H , which conflicts with the hypothesis $\text{Rot}_{\text{Fix}(h)}(H) \subset [-N, N]$. Therefore $|p/q| < N+1$. We have completed the proof. \square

The following Theorem is due to Franks [Fr1] when \mathbb{A} is a closed annulus and h has no wandering point, and it was improved by Le Calvez [Lec1] to the case where \mathbb{A} is an open annulus and h satisfies the intersection property:

Theorem 3.5. *Let $h \in \text{Homeo}_*(\mathbb{A})$ and H be a lift of h to \mathbb{R}^2 . Suppose that there exist two positively recurrent points of rotation numbers ν^- and ν^+ (eventually equal to $\pm\infty$) with $\nu^- < \nu^+$, and suppose that h satisfies the following intersection property: any simple closed curve of \mathbb{A} which is not null-homotopic meets its image by h . Then for any rational number $p/q \in]\nu^-, \nu^+[$ written in an irreducible way, there exists a periodic point of period q whose rotation number is p/q .*

4. EXTENSION OF THE LINKING NUMBER

In this section, we will first extend the notion of linking number defined in 1.4.5, and then state some properties about it.

4.1. Extension of the linking number for a positively recurrent point.

Recall that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on a closed oriented surface M of genus $g \geq 1$ and \tilde{F} is the time-one map of the lifted identity isotopy $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]}$ on the universal cover \tilde{M} of M . For every distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , by Lemma 1.2, we can choose an isotopy \tilde{I}_1 from $\text{Id}_{\tilde{M}}$ to \tilde{F} that fixes \tilde{a} and \tilde{b} .

Let us fix $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$ and consider an open disk $U \subset M \setminus \pi(\{\tilde{a}, \tilde{b}\})$ containing z . For every pair $(z', z'') \in U^2$, choose an oriented simple path $\gamma_{z', z''}$ in U from z' to z'' . Denote by $\tilde{\Phi}$ the lift of the first return map Φ :

$$\begin{aligned} \tilde{\Phi} : \pi^{-1}(\text{Rec}^+(F)) \cap \pi^{-1}(U) &\rightarrow \pi^{-1}(\text{Rec}^+(F)) \cap \pi^{-1}(U) \\ \tilde{z} &\mapsto \tilde{F}^{\tau(z)}(\tilde{z}), \end{aligned}$$

where $z = \pi(\tilde{z})$ and $\tau(z)$ is the first return time in U .

For any $\tilde{z} \in \pi^{-1}(U)$, write $U_{\tilde{z}}$ the connected component of $\pi^{-1}(U)$ that contains \tilde{z} . For every $j \geq 0$, recall that $\tau_j(z) = \sum_{i=0}^{j-1} \tau(\Phi^i(z))$. For every $n \geq 1$, consider the following curves in \tilde{M} :

$$\tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n = \tilde{I}_1^{\tau_n(z)}(\tilde{z}) \tilde{\gamma}_{\tilde{\Phi}^n(\tilde{z}), \tilde{z}_n},$$

where $\tilde{z}_n \in \pi^{-1}(\{z\}) \cap \tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$, and $\tilde{\gamma}_{\tilde{\Phi}^n(\tilde{z}), \tilde{z}_n}$ is the lift of $\gamma_{\Phi^n(z), z}$ in that is contained $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$. We can define the following infinite product (see 1.1):

$$\tilde{\Gamma}_{\tilde{I}_1, z}^n = \prod_{\pi(\tilde{z})=z} \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n.$$

In particular, when $z \in \text{Fix}(F)$, $\tilde{\Gamma}_{\tilde{I}_1, z}^1 = \prod_{\pi(\tilde{z})=z} \tilde{I}_1(\tilde{z})$.

When $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})} = \tilde{U}_{\tilde{z}}$, the curve $\tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n$ is a loop and hence $\tilde{\Gamma}_{\tilde{I}_1, z}^n$ is an infinite family of loops, that will be called a *multi-loop*. When $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})} \neq \tilde{U}_{\tilde{z}}$, the curve $\tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n$ is a compact path and

hence $\tilde{\Gamma}_{I_1, z}^n$ is an infinite family of paths (it can be seen as a family of proper paths, that means all of two ends of these paths going to ∞), that will be called a *multi-path*.

In the both cases, for every neighborhood \tilde{V} of ∞ , there are finitely many loops or paths $\tilde{\Gamma}_{I_1, \tilde{z}}^n$ that are not included in \tilde{V} . By adding the point ∞ at infinity, we get a multi-loop on the sphere $\mathbf{S} = \tilde{M} \sqcup \{\infty\}$.

In fact, $\tilde{\Gamma}_{I_1, z}^n$ can be seen as a multi-loop in the annulus $A_{\tilde{a}, \tilde{b}}$ with a finite homology. As a consequence, if $\tilde{\gamma}$ is a path from \tilde{a} to \tilde{b} , the intersection number $\tilde{\gamma} \wedge \tilde{\Gamma}_{I_1, z}^n$ is well defined and does not depend on $\tilde{\gamma}$. By Remark 1.4 and the properties of intersection number, the intersection number is also independent of the choice of the identity isotopy \tilde{I}_1 but depends on U . Moreover, observe that the path $(\prod_{i=0}^{n-1} \gamma_{\Phi^{n-i}(z), \Phi^{n-i-1}(z)})(\gamma_{\Phi^n(z), z})^{-1}$ is a loop in U , we have

$$(4.1.1) \quad \tilde{\gamma} \wedge \tilde{\Gamma}_{I_1, z}^n = \tilde{\gamma} \wedge \prod_{j=0}^{n-1} \tilde{\Gamma}_{I_1, \Phi^j(z)}^1 = \sum_{j=0}^{n-1} \tilde{\gamma} \wedge \tilde{\Gamma}_{I_1, \Phi^j(z)}^1.$$

For $n \geq 1$, we can define the functions

$$(4.1.2) \quad L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{I_1, z}^n = \sum_{j=0}^{n-1} L_1(\tilde{F}; \tilde{a}, \tilde{b}, \Phi^j(z)).$$

where $U \subset M \setminus \pi(\{\tilde{a}, \tilde{b}\})$. The last equation follows from Equation 4.1.1. The function L_n depends on U but not on the choice of $\gamma_{\Phi^n(z), z}$.

Definition 4.1. Fix $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$. Let us say that the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z) \in \mathbb{R}$ is defined, if

$$\lim_{k \rightarrow +\infty} \frac{L_{n_k}(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau_{n_k}(z)} = i(\tilde{F}; \tilde{a}, \tilde{b}, z)$$

for any subsequence $\{\Phi^{n_k}(z)\}_{k \geq 1}$ of $\{\Phi^n(z)\}_{n \geq 1}$ which converges to z .

Note here that the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ does not depend on U since if U and U' are open disks containing z , there exists a disk containing z that is contained in $U \cap U'$. In particular, when $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ always exists and is equal to $L_1(\tilde{F}; \tilde{a}, \tilde{b}, z)$.

Remark 4.2. When $z \in \text{Rec}^+(F) \setminus \text{Fix}_{\text{Cont}, I}(F)$, the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ depends on the choice of \tilde{a} and \tilde{b} if it exists. Indeed, consider the following smooth identity isotopy on \mathbb{R}^2 : $\tilde{I} = (\tilde{F}_t)_{t \in [0, 1]} : (x, y) \mapsto (x, y + t \sin(2\pi x))$. It induces an identity smooth isotopy $I = (F_t)_{t \in [0, 1]}$ on $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. Obviously $\text{Fix}(\tilde{F}) = \{(x, y) \mid x = k, x = k + 1/2, k \in \mathbb{Z}\}$ and $z = (1/4, 0) \in \mathbb{T}^2$ is a fixed point of F but not a contractible fixed point of F . Let $\tilde{a}_k = (k, 1/2) \in \mathbb{R}^2$ where $k \in \mathbb{Z}$. It is easy to see that $i(\tilde{F}; \tilde{a}_0, \tilde{a}_k, z) = k$ and $\pi(\tilde{a}_k) = \pi(\tilde{a}_{k'})$ where $k, k' \in \mathbb{Z}$.

4.2. Some properties of the linking number. Now we give some properties of the linking number we have defined.

For any $q \geq 1$, F^q is the time-one map of the identity isotopy $I^q = (F_t)_{t \in [0, q]}$ on M . By Lemma 8.1 in Appendix, a positively recurrent point of F is also a positively recurrent point of F^q , so we can define the linking number $i(\tilde{F}^q; \tilde{a}, \tilde{b}, z)$.

Proposition 4.3. *If $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists, then $i(\tilde{F}^q; \tilde{a}, \tilde{b}, z)$ exists for every $q \geq 1$ and $i(\tilde{F}^q; \tilde{a}, \tilde{b}, z) = qi(\tilde{F}; \tilde{a}, \tilde{b}, z)$.*

Proof. Let $\tilde{\gamma}$ be any simple path from \tilde{a} to \tilde{b} and \tilde{I}_1 be an isotopy that fixes \tilde{a} and \tilde{b} . We suppose that $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists. Let U be an open disk containing z . For every $q \geq 1$, write respectively $\tau'(z)$ and $\Phi'(z)$ for the first return time and the first return map of F^q in this proof. Recall that

$$\tau'_n(z) = \sum_{i=0}^{n-1} \tau'(\Phi'^i(z))$$

and

$$\tilde{\Gamma}_{\tilde{I}_1^q, \tilde{z}}^n = \tilde{I}_1^{q\tau'_n(z)}(\tilde{z})\tilde{\gamma}_{\tilde{\Phi}'^n(\tilde{z}), \tilde{z}_n}, \quad \tilde{\Gamma}_{\tilde{I}_1^q, z}^n = \prod_{\pi(\tilde{z})=z} \tilde{\Gamma}_{\tilde{I}_1^q, \tilde{z}}^n$$

where $\tilde{\Phi}'$ is the lift of Φ' to $\pi^{-1}(U)$, $\tilde{z}_n \in \pi^{-1}(\{z\}) \cap \tilde{U}_{\tilde{\Phi}'^n(\tilde{z})}$ and $\tilde{\gamma}_{\tilde{\Phi}'^n(\tilde{z}), \tilde{z}_n}$ is the lift of $\gamma_{\Phi'^n(z), z}$ that is in $\tilde{U}_{\tilde{\Phi}'^n(\tilde{z})}$.

We suppose that the subsequence $\{\Phi'^{n_k}(z)\}_{k \geq 1}$ converges to z . For every k , there is $n'_k \in \mathbb{N}$ such that $\tau'_{n'_k}(z) = q\tau'_{n_k}(z)$. By Definition 4.1, for any subsequence $\{\Phi'^{n_k}(z)\}_{k \geq 1}$ which converges to z , we have

$$\begin{aligned} \lim_{k \rightarrow +\infty} \frac{L_{n_k}(\tilde{F}^q; \tilde{a}, \tilde{b}, z)}{\tau'_{n_k}(z)} &= \lim_{k \rightarrow +\infty} \frac{\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1^q, z}^{n_k}}{\tau'_{n_k}(z)} \\ &= q \cdot \lim_{k \rightarrow +\infty} \frac{\tilde{\gamma} \wedge \prod_{\pi(\tilde{z})=z} \tilde{I}_1^{q\tau'_{n_k}(z)}(\tilde{z})\tilde{\gamma}_{\tilde{\Phi}'^{n_k}(\tilde{z}), \tilde{z}_{n_k}}}{q\tau'_{n_k}(z)} \\ &= q \cdot \lim_{k \rightarrow +\infty} \frac{L_{n'_k}(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau'_{n'_k}(z)} \\ &= qi(\tilde{F}; \tilde{a}, \tilde{b}, z). \end{aligned}$$

□

Proposition 4.4. *For every $\alpha \in G$, every distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , and every $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, if $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists, then $i(\tilde{F}; \alpha(\tilde{a}), \alpha(\tilde{b}), z)$ also exists and*

$$i(\tilde{F}; \alpha(\tilde{a}), \alpha(\tilde{b}), z) = i(\tilde{F}; \tilde{a}, \tilde{b}, z).$$

Proof. Let $\tilde{\gamma}$ be any simple path from \tilde{a} to \tilde{b} . Observe that the isotopy $\tilde{I}'_1 = \alpha \circ \tilde{I}_1 \circ \alpha^{-1}$ fixes $\alpha(\tilde{a})$ and $\alpha(\tilde{b})$, $\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n = \alpha(\tilde{\gamma}) \wedge \tilde{\Gamma}_{\tilde{I}'_1, \alpha(\tilde{z})}^n$ for every n . The proposition follows from Definition 4.1. □

Let H be an orientation preserving homeomorphism of M and \tilde{H} be a lift of H to \tilde{M} . Consider the time-one map $H \circ F \circ H^{-1}$ of the isotopy $I' = H \circ I \circ H^{-1}$ and write the time-one map of the identity isotopy \tilde{I}' as $\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}$, where \tilde{I}' is the lift of I' to \tilde{M} . Similarly to the Proposition 4.4, we have the following result:

Proposition 4.5. *For every distinct fixed points \tilde{a}, \tilde{b} of \tilde{F} and every $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, if $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists, then $i(\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}; \tilde{H}(\tilde{a}), \tilde{H}(\tilde{b}), H(z))$ also exists and*

$$i(\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}; \tilde{H}(\tilde{a}), \tilde{H}(\tilde{b}), H(z)) = i(\tilde{F}; \tilde{a}, \tilde{b}, z).$$

Proposition 4.6. *For every distinct fixed points \tilde{a}, \tilde{b} and \tilde{c} of \tilde{F} , and every $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}, \tilde{c}\})$, if two among the three linking numbers $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$, $i(\tilde{F}; \tilde{b}, \tilde{c}, z)$ and $i(\tilde{F}; \tilde{c}, \tilde{a}, z)$ exist, then the last one also exists and we have*

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) + i(\tilde{F}; \tilde{b}, \tilde{c}, z) + i(\tilde{F}; \tilde{c}, \tilde{a}, z) = 0.$$

Before proving Proposition 4.6, we introduce some notations and recall some results of the annulus.

If $\{\gamma_i\}_{1 \leq i \leq k}$ and $\{\gamma'_j\}_{1 \leq j \leq k'}$ are two finite families of loops or compact paths in $\mathbf{S} = \tilde{M} \cup \{\infty\}$ such that $\prod_{i=1}^k \gamma_i$ and $\prod_{j=1}^{k'} \gamma'_j$ are well defined (in the concatenation sense) (see 1.1) and the algebraic intersection number $\left(\prod_{i=1}^k \gamma_i\right) \wedge \left(\prod_{j=1}^{k'} \gamma'_j\right)$ is well defined (see 1.2), then we formally write

$$\left(\prod_{i=1}^k \gamma_i\right) \wedge \left(\prod_{j=1}^{k'} \gamma'_j\right) = \sum_{i,j} \gamma_i \wedge \gamma'_j.$$

Recall that $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ is the open annulus and $T : (x, y) \mapsto (x+1, y)$ is the generator of the covering transformation group. If $I = (h_t)_{t \in [0,1]}$ with $h_0 = h_1 = \text{Id}_{\mathbb{A}}$ is a loop in $\text{Homeo}_*(\mathbb{A})$, write $[I]_1 \in \pi_1(\text{Homeo}_*(\mathbb{A}))$ for the homotopy class of I . Recall that $\pi_1(\text{Homeo}_*(\mathbb{A})) \simeq \mathbb{Z}$. Therefore, we may write $\pi_1(\text{Homeo}_*(\mathbb{A})) = \bigcup_{k \in \mathbb{Z}} \mathcal{C}_k$ where \mathcal{C}_k is the class which satisfies that for every $[I]_1 \in \mathcal{C}_k$, any lift \tilde{I} of I to the universal covering space $\tilde{\mathbb{A}}$ satisfies $\tilde{h}_1(\tilde{z}) - \tilde{h}_0(\tilde{z}) = T^k(\tilde{z})$ for every $\tilde{z} \in \tilde{\mathbb{A}}$.

Proof of Proposition 4.6. Suppose that $\tilde{\gamma}_1, \tilde{\gamma}_2$ and $\tilde{\gamma}_3$ are oriented simple paths from \tilde{a} to \tilde{b} , \tilde{b} to \tilde{c} and \tilde{c} to \tilde{a} , respectively. We choose isotopies \tilde{I}_j ($j = 0, 1, 2, 3$) such that \tilde{I}_1 fixes \tilde{a}, \tilde{b} and ∞ , \tilde{I}_2 fixes \tilde{b}, \tilde{c} and ∞ , \tilde{I}_3 fixes \tilde{c}, \tilde{a} and ∞ , and \tilde{I}_0 fixes \tilde{a}, \tilde{b} and \tilde{c} .

For every $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}, \tilde{c}\})$, every lift \tilde{z} of z , every $j \in \{1, 2, 3\}$ and every $n \geq 1$, the path $\tilde{I}_0^n(\tilde{z})(\tilde{I}_j^n(\tilde{z}))^{-1}$ is a loop where $(\tilde{I}_j^n(\tilde{z}))^{-1}$ is the inverse of the path $\tilde{I}_j^n(\tilde{z})$. We claim that

$$(4.2.1) \quad \tilde{\gamma}_j \wedge \left(\tilde{I}_0^n(\tilde{z})(\tilde{I}_j^n(\tilde{z}))^{-1}\right) = \tilde{\gamma}_j \wedge \tilde{I}_0^n(\tilde{z}) - \tilde{\gamma}_j \wedge \tilde{I}_j^n(\tilde{z}) = n \cdot (\tilde{\gamma}_j \wedge \tilde{I}_0(\infty)).$$

Indeed, let \mathbb{A}_j ($j = 1, 2, 3$) be respectively $\mathbf{S} \setminus \{\tilde{a}, \tilde{b}\}$, $\mathbf{S} \setminus \{\tilde{b}, \tilde{c}\}$ and $\mathbf{S} \setminus \{\tilde{c}, \tilde{a}\}$. For every $n \in \mathbb{N}$, considering the loops $\tilde{I}_j^{-n} \tilde{I}_0^n \subset \text{Homeo}_*(\mathbb{A}_j)$ (see 1.1.1) where \tilde{I}_j^{-1} is the inverse of

\tilde{I}_j , we have $[\tilde{I}_j^{-n} \tilde{I}_0^n]_1 \in \mathcal{C}_{n \cdot (\tilde{\gamma}_j \wedge \tilde{I}_0(\infty))}^j$ ($j = 1, 2, 3$) where \mathcal{C}_k^j is a class in $\pi_1(\text{Homeo}_*(\mathbb{A}_j))$. Observing that $(\tilde{I}_j^{-n} \tilde{I}_0^n)(\tilde{z}) = \tilde{I}_0^n(\tilde{z})(\tilde{I}_j^n(\tilde{z}))^{-1}$, the claim 4.2.1 follows.

In the case where $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}, \tilde{c}\})$, for every lift \tilde{z} of z , we have

$$\tilde{\gamma}_j \wedge \tilde{I}_0(\tilde{z}) - \tilde{\gamma}_j \wedge \tilde{I}_j(\tilde{z}) = \tilde{\gamma}_j \wedge \tilde{I}_0(\infty) \quad (j = 1, 2, 3).$$

Write C_z for the set of points $\tilde{z} \in \pi^{-1}(\{z\})$ such that $\tilde{I}_j(\tilde{z}) \cap \bigcup_{j'=1}^3 \tilde{\gamma}_{j'} \neq \emptyset$ for every j .

As all \tilde{I}_j fix ∞ , we know that C_z is finite.

Recall that

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma}_1 \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^1, \quad i(\tilde{F}; \tilde{b}, \tilde{c}, z) = \tilde{\gamma}_2 \wedge \tilde{\Gamma}_{\tilde{I}_2, z}^1 \quad \text{and} \quad i(\tilde{F}; \tilde{c}, \tilde{a}, z) = \tilde{\gamma}_3 \wedge \tilde{\Gamma}_{\tilde{I}_3, z}^1$$

where

$$\tilde{\Gamma}_{\tilde{I}_j, z}^1 = \prod_{\pi(\tilde{z})=z} \tilde{I}_j(\tilde{z}) \quad (j = 1, 2, 3).$$

Observe that

$$\sum_{j=1}^3 \sum_{\tilde{z} \in C_z} \tilde{\gamma}_j \wedge \tilde{I}_0(\tilde{z}) = \sum_{\tilde{z} \in C_z} \sum_{j=1}^3 \tilde{\gamma}_j \wedge \tilde{I}_0(\tilde{z}) = 0, \quad \sum_{j=1}^3 \tilde{\gamma}_j \wedge \tilde{I}_0(\infty) = 0$$

and

$$\tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, z}^1 = \tilde{\gamma}_j \wedge \prod_{\pi(\tilde{z})=z} \tilde{I}_j(\tilde{z}) = \sum_{\tilde{z} \in C_z} \tilde{\gamma}_j \wedge \tilde{I}_j(\tilde{z}) \quad (j = 1, 2, 3).$$

We get

$$\begin{aligned} & i(\tilde{F}; \tilde{a}, \tilde{b}, z) + i(\tilde{F}; \tilde{b}, \tilde{c}, z) + i(\tilde{F}; \tilde{c}, \tilde{a}, z) \\ &= \sum_{j=1}^3 \left(\tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, z}^1 \right) \\ &= - \sum_{j=1}^3 \sum_{\tilde{z} \in C_z} \left(\tilde{\gamma}_j \wedge \tilde{I}_0(\tilde{z}) - \tilde{\gamma}_j \wedge \tilde{I}_j(\tilde{z}) \right) \\ &= - \sum_{\tilde{z} \in C_z} \sum_{j=1}^3 \tilde{\gamma}_j \wedge \tilde{I}_0(\infty) \\ &= 0. \end{aligned}$$

Hence we have proved the proposition in this case.

In the case where $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$, recall that

$$\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n = \tilde{I}_j^{\tau_n(z)}(\tilde{z}) \tilde{\gamma}_{\tilde{\Phi}^n(\tilde{z}), \tilde{z}_n} \quad (0 \leq j \leq 3),$$

where $\tilde{z}_n \in \pi^{-1}(\{z\}) \cap \tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$ and $\tilde{\gamma}_{\tilde{\Phi}^n(\tilde{z}), \tilde{z}_n}$ is the lift of $\gamma_{\Phi^n(z), z}$ in $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$. For every $1 \leq j \leq 3$, we have $\tilde{\Gamma}_{\tilde{I}_0, \tilde{z}}^n (\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n)^{-1}$ is a loop where $(\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n)^{-1}$ is the inverse of the path $\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n$.

Therefore, for every lift \tilde{z} of z and $n \geq 1$, we have

$$\tilde{\gamma}_j \wedge \left(\tilde{\Gamma}_{\tilde{I}_0, \tilde{z}}^n (\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n)^{-1} \right) = \tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_0, \tilde{z}}^n - \tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n = \tau_n(z) \cdot (\tilde{\gamma}_j \wedge \tilde{I}_0(\infty)) \quad (j = 1, 2, 3).$$

For every n , write C_z^n for the set of points $z \in \pi^{-1}(\{z\})$ such that $\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n \cap \bigcup_{j=1}^3 \tilde{\gamma}_j \neq \emptyset$.

Here again, we know that C_z^n is finite.

Recall that

$$L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma}_1 \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n, \quad L_n(\tilde{F}; \tilde{b}, \tilde{c}, z) = \tilde{\gamma}_2 \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n \quad \text{and} \quad L_n(\tilde{F}; \tilde{c}, \tilde{a}, z) = \tilde{\gamma}_3 \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n$$

where

$$\tilde{\Gamma}_{\tilde{I}_1, z}^n = \prod_{\pi(\tilde{z})=z} \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n.$$

Then for any subsequence $\{\Phi^{n_k}(z)\}_{k \geq 1}$ which converges to z , similarly to the fixed point case, we get

$$\begin{aligned} (4.2.2) \quad & \frac{L_{n_k}(\tilde{F}; \tilde{a}, \tilde{b}, z) + L_{n_k}(\tilde{F}; \tilde{b}, \tilde{c}, z) + L_{n_k}(\tilde{F}; \tilde{c}, \tilde{a}, z)}{\tau_{n_k}(z)} \\ &= \frac{1}{\tau_{n_k}(z)} \sum_{j=1}^3 \left(\tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, z}^{n_k} \right) \\ &= -\frac{1}{\tau_{n_k}(z)} \sum_{j=1}^3 \sum_{\tilde{z} \in C_z^{n_k}} \left(\tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_0, \tilde{z}}^{n_k} - \tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^{n_k} \right) \\ &= - \sum_{\tilde{z} \in C_z^{n_k}} \sum_{j=1}^3 \tilde{\gamma}_j \wedge \tilde{I}_0(\infty) \\ &= 0. \end{aligned}$$

Letting $k \rightarrow +\infty$ in Equation 4.2.2, we have completed the proposition. \square

5. BOUNDEDNESS AND EXISTENCE OF THE LINKING NUMBER

This section is divided into two parts. In the first part, we study the boundedness of the linking number when it exists. In the second part, we study the existence and boundedness of the linking number if the map F preserves a measure on M . The tools we will use are Franks' Lemma and Birkhoff Ergodic Theorem.

5.1. Boundedness. In this section, let \tilde{a} and \tilde{b} be two distinct fixed points of \tilde{F} . We suppose that I satisfies WB-property at \tilde{a} and \tilde{b} . By Lemma 1.8, there is a positive number $N_{\tilde{a}, \tilde{b}}$ such that $\text{Rot}_{\text{Fix}(\tilde{F}_{\tilde{a}, \tilde{b}})}(\tilde{F}_{\tilde{a}, \tilde{b}}) \subset [-N_{\tilde{a}, \tilde{b}}, N_{\tilde{a}, \tilde{b}}]$.

Fix an isotopy \tilde{I}_1 from $\text{Id}_{\tilde{M}}$ to \tilde{F} which fixes \tilde{a} and \tilde{b} . Let $\tilde{\gamma}$ be any oriented path in \tilde{M} from \tilde{a} to \tilde{b} . Fix an open disk \tilde{W} that contains ∞ and is disjoint from $\tilde{\gamma}$. We choose an open disk $\tilde{V} \subset \tilde{W}$ that contains ∞ such that for every $\tilde{z} \in \tilde{V}$, we have $\tilde{I}_1(\tilde{z}) \subset \tilde{W}$.

Observe that if $\widehat{\infty}$ is a given lift of ∞ in $\widehat{A}_{\tilde{a}, \tilde{b}}$, if \widehat{W} (resp. \widehat{V}) is the connected component of $\pi^{-1}(\widetilde{W})$ (resp. $\pi^{-1}(\widetilde{V})$) that contains $\widehat{\infty}$, then we have $\widehat{F}_{\tilde{a}, \tilde{b}}(\widehat{V}) \subset \widehat{W}$, which implies that \widehat{V} is free for every other lift $\widehat{F}_{\tilde{a}, \tilde{b}} \circ T_{\tilde{a}, \tilde{b}}^k$, where $k \in \mathbb{Z} \setminus \{0\}$. Let A^c denote the complement of a set A . For every $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}\})$, write $X_z = \pi^{-1}(\{z\}) \cap (\widetilde{V} \cap \widetilde{F}_{\tilde{a}, \tilde{b}}^{-1}(\widetilde{V}))^c$. Observe that there exists $K_{\tilde{a}, \tilde{b}} \in \mathbb{N}$ such that $\sharp X_z \leq K_{\tilde{a}, \tilde{b}}$ for every $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}\})$.

In the case where $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$, we choose an open disk U that contains z and is free for F . As the value $i(\widetilde{F}; \tilde{a}, \tilde{b}, z)$ depends neither on $\tilde{\gamma}$ nor on U , we can always suppose that $\tilde{\gamma} \cap \pi^{-1}(U) = \emptyset$ by perturbing $\tilde{\gamma}$ a little and shrinking U if necessary. For every $n \geq 1$, write

$$X_z^n = \pi^{-1}(\{z, F(z), \dots, F^{\tau_n(z)-1}(z)\}) \cap (\widetilde{V} \cap \widetilde{F}_{\tilde{a}, \tilde{b}}^{-1}(\widetilde{V}))^c.$$

Observe that $\sharp X_z^n \leq \tau_n(z) K_{\tilde{a}, \tilde{b}}$.

The following result is the main proposition of this section.

Proposition 5.1. *The following two statements hold:*

- If $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, we have $|i(\widetilde{F}; \tilde{a}, \tilde{b}, z)| < K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$.
- If $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$ and $i(\widetilde{F}; \tilde{a}, \tilde{b}, z)$ is defined, then $|i(\widetilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a}, \tilde{b}} K_U$, where $K_U \in \mathbb{N}$ depends only on U .

In order to prove Proposition 5.1, we consider two cases: the fixed point case and the non-fixed point case. The first case is more easy to deal with and the second case is a little more complicated, but the ideas are similar.

The fixed point case.

When $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, then $\tau(z) = 1$ and $i(\widetilde{F}; \tilde{a}, \tilde{b}, z) = L_1(\widetilde{F}; \tilde{a}, \tilde{b}, z)$, we have the following results.

Lemma 5.2. *If $z \in \text{Fix}_{\text{Cont}, I}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, then $|i(\widetilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a}, \tilde{b}} N_{\tilde{a}, \tilde{b}}$.*

Proof. By Definition 4.1 and Lemma 1.6, we have

$$i(\widetilde{F}; \tilde{a}, \tilde{b}, z) = \sum_{\pi(\tilde{z})=z} \rho_{A_{\tilde{a}, \tilde{b}}, \widehat{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}) = \sum_{\tilde{z} \in X_z} \rho_{A_{\tilde{a}, \tilde{b}}, \widehat{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}).$$

The lemma follows from the fact that $\sharp X_z \leq K_{\tilde{a}, \tilde{b}}$ and that $\text{Rot}_{\text{Fix}(\widetilde{F}_{\tilde{a}, \tilde{b}})}(\widehat{F}_{\tilde{a}, \tilde{b}}) \subset [-N_{\tilde{a}, \tilde{b}}, N_{\tilde{a}, \tilde{b}}]$. \square

Lemma 5.3. *If $z \in \text{Fix}(F) \setminus \text{Fix}_{\text{Cont}, I}(F)$, then $|i(\widetilde{F}; \tilde{a}, \tilde{b}, z)| < K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$.*

Proof. We have

$$i(\widetilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \widetilde{\Gamma}_{I_1, z}^1 = \sum_{\tilde{z} \in X_z} \tilde{\gamma} \wedge \widetilde{I}_1(\tilde{z}).$$

Observe that if $\tilde{z} \in X_z$, then the trajectory of $\tilde{I}_1(\tilde{z})$ is not included in \tilde{V} . Therefore we can write the multi-path $\prod_{\tilde{z} \in X_z} \tilde{I}_1(\tilde{z})$ as finitely many sub-paths:

$$\prod_{\tilde{z} \in X_z} \tilde{I}_1(\tilde{z}) = \prod_{1 \leq i \leq P(z)} \tilde{\Gamma}_i(z),$$

where

$$\tilde{\Gamma}_i(z) = \prod_{0 \leq j < m^i(z)} \tilde{I}_1(\tilde{F}_{\tilde{a}, \tilde{b}}^j(\tilde{z}_i))$$

is a path with $\tilde{z}_i \in X_z \cap \tilde{V}$, $\tilde{F}_{\tilde{a}, \tilde{b}}^j(\tilde{z}_i) \in X_z \cap \tilde{V}^c$ for $1 \leq j < m^i$ and $\tilde{F}_{\tilde{a}, \tilde{b}}^{m^i}(\tilde{z}_i) \in \tilde{V}$. For every i , we get a periodic disk chain $C_i = (\{\tilde{V}, \tilde{V}\}, \{m^i\})$ whose length $l(C_i)$ is equal to m^i (see Section 3).

Obviously, $\sum_i m^i \leq K_{\tilde{a}, \tilde{b}}$. Let $k^i(z) = \tilde{\gamma} \wedge \tilde{\Gamma}_i$. We have $i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{I_1, z}^1 = \sum_i k^i$. Therefore, to get the lemma, it is sufficient to prove that $|k^i| < m^i(N_{\tilde{a}, \tilde{b}} + 1)$.

For every i , the path $\tilde{\Gamma}_i$ is lifted to a path from a point $\tilde{z}_i \in \tilde{V}$ to $\tilde{F}_{\tilde{a}, \tilde{b}}^{m^i}(\tilde{z}_i) \in T_{\tilde{a}, \tilde{b}}^{k^i}(\tilde{V})$ and hence we get a lift $\tilde{C}_i = (\{\tilde{V}, T_{\tilde{a}, \tilde{b}}^{k^i}(\tilde{V})\}, \{m^i\})$ of C_i for $\tilde{F}_{\tilde{a}, \tilde{b}}$ with width $w(\tilde{F}_{\tilde{a}, \tilde{b}}; \tilde{C}_i) = k^i$. By the construction of \tilde{V} , replacing \mathbb{A} by $\tilde{A}_{\tilde{a}, \tilde{b}}$, h by $\tilde{F}_{\tilde{a}, \tilde{b}}$, H by $\tilde{F}_{\tilde{a}, \tilde{b}}$, D by \tilde{V} and C by C_i in Lemma 3.4, we get $|k^i| < m^i(N_{\tilde{a}, \tilde{b}} + 1)$. We have completed the proof. \square

The non-fixed point case.

Let $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$ and U be an open free disk for F that contains z . Recall that, for every lift \tilde{z} of z and every $n \geq 0$, there is a unique connected component $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$ of $\pi^{-1}(U)$ such that $\tilde{\Phi}^n(\tilde{z}) \in \tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$ and a unique $\alpha_{z, n} \in G$ such that $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})} = \alpha_{z, n}(\tilde{U}_{\tilde{z}})$. For convenience, we define

$$\tilde{F}_{\tilde{a}, \tilde{b}}^*(\tilde{z}') = \begin{cases} \tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}') & \text{if } \pi(\tilde{z}') \in \{z, \dots, F^{\tau_n(z)-2}(z)\}; \\ \alpha_{z, n}(\tilde{z}) & \text{if } \pi(\tilde{z}') = F^{\tau_n(z)-1}(z) \text{ and } \tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}') \in \tilde{U}_{\alpha_{z, n}(\tilde{z})}. \end{cases}$$

and

$$\tilde{I}_1^*(\tilde{z}') = \begin{cases} \tilde{I}_1(\tilde{z}') & \text{if } \pi(\tilde{z}') \in \{z, \dots, F^{\tau_n(z)-2}(z)\}; \\ \tilde{I}_1(\tilde{z}') \tilde{\gamma}_{\tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}'), \alpha_{z, n}(\tilde{z})} & \text{if } \pi(\tilde{z}') = F^{\tau_n(z)-1}(z) \text{ and } \tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}') \in \tilde{U}_{\alpha_{z, n}(\tilde{z})}, \end{cases}$$

where $\tilde{\gamma}_{\tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}'), \alpha_{z, n}(\tilde{z})}$ is the lift of $\gamma_{\Phi^n(z), z}$ that is in $\tilde{U}_{\alpha_{z, n}(\tilde{z})}$.

We have to consider two cases: $\alpha_{z, n} = e$ and $\alpha_{z, n} \neq e$. First, we consider the case where $\alpha_{z, n} \neq e$. We have the following lemma.

Lemma 5.4. *If $\alpha_{z, n} \neq e$, then $|L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)| < \tau_n(z)K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$.*

Proof. In this case, the curve $\tilde{\Gamma}_{I_1, z}^n$ is a multi-path in \tilde{M} . By the definition of $L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)$, we have

$$L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{I_1, z}^n = \sum_{\tilde{z}' \in X_z^n} \tilde{\gamma} \wedge \tilde{I}_1^*(\tilde{z}').$$

We can write the multi-path

$$(5.1.1) \quad \prod_{\tilde{z}' \in X_z^n} \tilde{I}_1^*(\tilde{z}') = \prod_{1 \leq i \leq P_n(z)} \tilde{\Gamma}_i^n(z),$$

where

$$(5.1.2) \quad \tilde{\Gamma}_i^n(z) = \prod_{0 \leq j < m_n^i(z)} \tilde{I}_1^*(\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i))$$

is a path with $\tilde{z}_i \in X_z^n \cap \tilde{V}$, $\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i) \in X_z^n \cap \tilde{V}^c$ for $1 \leq j < m_n^i$ and $\tilde{F}_{\tilde{a}, \tilde{b}}^{*m_n^i}(\tilde{z}_i) \in \tilde{V}$. Hence, for every i , we get a periodic disk chain C_i that satisfies the hypothesis of Lemma 3.4 with length m_n^i . When we lift the path $\tilde{\Gamma}_i^n$, we can get a lift of C_i for $\hat{F}_{\tilde{a}, \tilde{b}}$ with width k_n^i .

Obviously, we have $\sum_i m_n^i < \tau_n K_{\tilde{a}, \tilde{b}}$. Let $k_n^i(z) = \tilde{\gamma} \wedge \tilde{\Gamma}_i^n$. Hence $L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_i k_n^i$. It is sufficient to prove that $|k_n^i| < m_n^i(N_{\tilde{a}, \tilde{b}} + 1)$.

Similarly to the proof of Lemma 5.3, replacing \mathbb{A} by $\tilde{A}_{\tilde{a}, \tilde{b}}$, h by $\tilde{F}_{\tilde{a}, \tilde{b}}$, H by $\hat{F}_{\tilde{a}, \tilde{b}}$, D by \tilde{V} and C by C_i in Lemma 3.4, we get $|k_n^i| < m_n^i(N_{\tilde{a}, \tilde{b}} + 1)$. This proves the first case. \square

As a consequence, we have the following proposition.

Proposition 5.5. *We suppose that $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ and $\rho_{M, I}(z)$ exist, then*

$$|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1) \quad \text{if} \quad \rho_{M, I}(z) \neq 0.$$

Proof. If $z \in \text{Fix}(F)$ and $\rho_{M, I}(z) \neq 0$, then z is not a contractible fixed point and the conclusion follows from Lemma 5.3. Suppose now that $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$ and $U \subset M \setminus \text{Fix}(F)$ is a free open disk containing z . If $\rho_{M, I}(z) \neq 0$, then there exists a positive number N such that $\alpha_{z, n} \neq e$ when $n \geq N$ (see 1.3.2). In that case, the conclusion follows from Lemma 5.4. \square

Let us study the case where $\alpha_{z, n} = e$.

Lemma 5.6. *There exists a positive integer K_U which depends on U such that*

$$|L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq \tau_n(z) K_{\tilde{a}, \tilde{b}} K_U \quad \text{if} \quad \alpha_{z, n} = e.$$

Before proving Lemma 5.6, we require the following lemma.

Lemma 5.7. *Let \tilde{U} be any connected component of $\pi^{-1}(U)$ in \tilde{V}^c . If*

$$\text{Rot}_{\tilde{U}}(\hat{F}_{\tilde{a}, \tilde{b}}) \not\subset]-(N_{\tilde{a}, \tilde{b}} + 1), N_{\tilde{a}, \tilde{b}} + 1[,$$

then we have

- (1) $\alpha_{z', n} = e$ for all $z' \in \text{Rec}^+(F) \cap U$ and all $n \geq 1$;
- (2) $\bigcup_{k \geq 1} \tilde{F}^k(\pi^{-1}(\text{Rec}^+(F)) \cap \tilde{U}) \subset \tilde{V}^c$;
- (3) $\text{Rot}_{\tilde{U}}(\hat{F}_{\tilde{a}, \tilde{b}}) \subset]l, l + 1[$ for some integer l with $l \geq N_{\tilde{a}, \tilde{b}} + 1$ or $l \leq -(N_{\tilde{a}, \tilde{b}} + 2)$ where l depends on \tilde{U} .

Let us prove now Lemma 5.6 supposing Lemma 5.7 whose proof will be given later.

Proof of Lemma 5.6. As $\alpha_{z,n} = e$, the curve $\tilde{\Gamma}_{I_1,z}^n$ is a multi-loop in \tilde{M} . Let $p_n(\tilde{z}) = \tilde{\gamma} \wedge \tilde{\Gamma}_{I_1,\tilde{z}}^n$ where $\tilde{z} \in \pi^{-1}(z)$. Obviously, $p_n(\tilde{z})/\tau_n(z) \in \text{Rot}_{\tilde{U}_{\tilde{z}}}(\tilde{F}_{\tilde{a},\tilde{b}})$.

Let us first analyze the possible cases that need to be considered in the proof. The set X_z^n maybe contain a “whole orbit” of some lift \tilde{z} of z , that means $\tilde{F}^j(\tilde{z}) \in X_z^n$ for all $0 \leq j < \tau_n(z)$, or a “partial orbit” of \tilde{z} . In the case where a “partial orbit” of \tilde{z} is contained in X_z^n , similarly to the proof of Lemma 5.3, we can get a periodic disk chain of $\tilde{F}_{\tilde{a},\tilde{b}}$ that satisfies the hypothesis of Lemma 3.4 and hence we can estimate the intersection number of $\tilde{\gamma}$ and the path on which the “partial orbit” of \tilde{z} lies. In the case where the “whole orbit” of \tilde{z} is contained in X_z^n , we can use Lemma 5.7 to get either $|p_n(\tilde{z})/\tau_n(z)| < N_{\tilde{a},\tilde{b}} + 1$, or $l < p_n(\tilde{z})/\tau_n(z) < l + 1$ where $l \in \mathbb{Z}$ depends on \tilde{U} and satisfies $l \geq N_{\tilde{a},\tilde{b}} + 1$ or $l \leq -(N_{\tilde{a},\tilde{b}} + 2)$. Finally, we only need to sum the intersection numbers of all the cases above.

Let us begin the rigorous proof. Write

$$S_z^n = \{\tilde{z} \in \pi^{-1}(z) \mid \tilde{F}^j(\tilde{z}) \in \tilde{V}^c \text{ for all } 0 \leq j < \tau_n(z)\}$$

and

$$Y_z^n = \{\tilde{F}^j(\tilde{z}) \mid \tilde{z} \in S_z^n, 0 \leq j < \tau_n(z)\}.$$

As before, we write

$$L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{I_1,z}^n = \sum_{\tilde{z}' \in X_z^n} \tilde{\gamma} \wedge \tilde{I}_1^n(\tilde{z}').$$

We can write the multi-path as follows

$$(5.1.3) \quad \prod_{\tilde{z}' \in X_z^n} \tilde{I}_1^*(\tilde{z}') = \prod_{\tilde{z}' \in Y_z^n} \tilde{I}_1^*(\tilde{z}') \cdot \prod_{\tilde{z}' \in X_z^n \setminus Y_z^n} \tilde{I}_1^*(\tilde{z}') = \prod_{1 \leq i \leq P'_n(z)} \tilde{\Gamma}_i^n(z) \cdot \prod_{P'_n(z) < i \leq P_n(z)} \tilde{\Gamma}_i^n(z),$$

where

$$(5.1.4) \quad \tilde{\Gamma}_i^n(z) = \tilde{\Gamma}_{I_1,\tilde{z}_i}^n = \prod_{0 \leq j < m_n^i(z)} \tilde{I}_1^*(\tilde{F}_{\tilde{a},\tilde{b}}^{*j}(\tilde{z}_i))$$

for $1 \leq i \leq P'_n$ with $\tilde{z}_i \in S_z^n$ and $m_n^i = \tau_n$; and

$$(5.1.5) \quad \tilde{\Gamma}_i^n(z) = \prod_{0 \leq j < m_n^i(z)} \tilde{I}_1^*(\tilde{F}_{\tilde{a},\tilde{b}}^{*j}(\tilde{z}_i))$$

for $P'_n < i \leq P_n$ with $\tilde{z}_i \in X_z^n \cap \tilde{V}$, $\tilde{F}_{\tilde{a},\tilde{b}}^{*j}(\tilde{z}_i) \in X_z^n \cap \tilde{V}^c$ for $1 \leq j < m_n^i$ and $\tilde{F}_{\tilde{a},\tilde{b}}^{*m_n^i}(\tilde{z}_i) \in \tilde{V}$.

Obviously, $\sum_i m_n^i \leq \tau_n(z) K_{\tilde{a},\tilde{b}}$. Let $k_n^i(z) = \tilde{\gamma} \wedge \tilde{\Gamma}_i^n$. Hence $L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_i k_n^i$. To prove Lemma 5.6, it is sufficient to prove that there exists a positive integer K_U which depends only on U such that $|k_n^i| \leq m_n^i K_U$.

When $1 \leq i \leq P'_n$, by Lemma 5.7 and the fact that $P'_n \leq K_{\tilde{a}, \tilde{b}}$, there exists a positive integer r that depends on U such that $\text{Rot}_{\tilde{U}_{\tilde{z}_i}}(\hat{F}_{\tilde{a}, \tilde{b}}) \subset [-r, r]$. Observing that $k_n^i = p_n(\tilde{z}_i) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}_i}^n$, $m_n^i = \tau_n$, and $k_n^i/m_n^i = p_n(\tilde{z}_i)/\tau_n(z) \in \text{Rot}_{\tilde{U}_{\tilde{z}_i}}(\hat{F}_{\tilde{a}, \tilde{b}})$, we have $|k_n^i| \leq m_n^i r$.

When $P'_n < i \leq P_n$, similarly to the proof of Lemma 5.3, we can get $|k_n^i| < m_n^i(N_{\tilde{a}, \tilde{b}} + 1)$.

Let $K_U = \max\{N_{\tilde{a}, \tilde{b}} + 1, r\}$. We have $|k_n^i| \leq m_n^i K_U$ for every $1 \leq i \leq P_n$ and hence

$$|L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)| = \left| \sum_i k_n^i \right| \leq \tau_n(z) K_{\tilde{a}, \tilde{b}} K_U.$$

We have completed the proof. \square

Proof of Lemma 5.7. (1). Suppose that there is a point $z' \in \text{Rec}^+(F) \cap U$ and some $n_0 \geq 1$ such that $\alpha_{z', n_0} \neq e$. Let \tilde{z}' be the lift of z' that is in \tilde{U} . Similarly to the proof of Lemma 5.4, we can find a path

$$\tilde{\Gamma}_i^{n_0}(z') = \prod_{0 \leq j < m_{n_0}^i(z')} \tilde{I}_1^*(\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i))$$

which satisfies $\tilde{z}_i \in X_{z'}^{n_0} \cap \tilde{V}$, $\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i) \in X_{z'}^{n_0} \cap \tilde{V}^c$ for all $1 \leq j < m_{n_0}^i$, $\tilde{z}' = \tilde{F}_{\tilde{a}, \tilde{b}}^{*j_0}(\tilde{z}_i)$ for some $1 \leq j_0 < m_{n_0}^i$, and $\tilde{F}_{\tilde{a}, \tilde{b}}^{*m_{n_0}^i}(\tilde{z}_i) \in \tilde{V}$. Hence, we get a periodic disk chain C' that contains \tilde{U} as an element and satisfies the hypothesis of Lemma 3.4. Replacing \mathbb{A} by $\tilde{A}_{\tilde{a}, \tilde{b}}$, h by $\tilde{F}_{\tilde{a}, \tilde{b}}$, H by $\hat{F}_{\tilde{a}, \tilde{b}}$, D by \tilde{V} and C by C' in Lemma 3.4 (the second conclusion), we get $\text{Rot}_{\tilde{U}}(\hat{F}_{\tilde{a}, \tilde{b}}) \subset] - (N_{\tilde{a}, \tilde{b}} + 1), N_{\tilde{a}, \tilde{b}} + 1[$. We have a contradiction.

(2). Suppose that there is a point $\tilde{z}' \in \pi^{-1}(z') \cap \tilde{U}$ where $z' \in \text{Rec}^+(F)$ and an integer $n_0 \geq 1$ such that $\tilde{F}^{n_0}(\tilde{z}') \in \tilde{V}$. By (1), it is sufficient to consider the case where $\alpha_{z', n} = e$ for all $n \geq 1$, that means, $\tilde{F}^{\tau_n(z')}(\tilde{z}') \in \tilde{U}$ for all $n \geq 1$. We choose a positive integer n_1 large enough such that $\tau_{n_1}(z') > n_0$. We have $\tilde{F}^{\tau_{n_1}(z') - n_0}(\tilde{F}^{n_0}(\tilde{z}')) \in \tilde{U}$. Then we get $\tilde{F}^{\tau_{n_1}(z') - n_0}(\tilde{V}) \cap \tilde{U} \neq \emptyset$ and $\tilde{F}^{n_0}(\tilde{U}) \cap \tilde{V} \neq \emptyset$. Therefore, the disk chain $(\{\tilde{V}, \tilde{U}, \tilde{V}\}, \{\tau_{n_1}(z') - n_0, n_0\})$ is a periodic disk chain that satisfies the hypothesis of Lemma 3.4. Applying Lemma 3.4 again, we get $\text{Rot}_{\tilde{U}}(\hat{F}_{\tilde{a}, \tilde{b}}) \subset] - (N_{\tilde{a}, \tilde{b}} + 1), N_{\tilde{a}, \tilde{b}} + 1[$. It is still a contradiction.

(3). This follows from Lemma 3.4 immediately. \square

Proof of Proposition 5.1. This follows from Lemma 5.2, Lemma 5.3, Lemma 5.4 and Lemma 5.6. \square

In the end of this section, we study the boundedness in the case where the time-one map F of I satisfies some differential conditions.

Proposition 5.8. *For any two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , if F and F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$, then there exists $N \in \mathbb{R}$ such that $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq N$ if $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists.*

Proof. We make a proof by contradiction. If it is not true, without loss of generality, we suppose that there is a sequence $\{z_k\}_{k \geq 1} \subset \text{Rec}^+(F)$ such that $\lim_{k \rightarrow +\infty} i(\tilde{F}; \tilde{a}, \tilde{b}, z_k) = +\infty$. By the proof of Lemma 5.6 and the conclusion (1) of Lemma 5.7, we have $\alpha_{z_k, n} = e$ for every $n \geq 1$ when k is large enough. Hence $\tilde{z}_k \in \text{Rec}^+(\tilde{F}) \setminus \text{Fix}(\tilde{F})$ when k is large enough where $\tilde{z}_k \in \pi^{-1}(z_k)$. By the proof of Lemma 5.6 and the conclusion (2) of Lemma 5.7, we only need consider the lifts \tilde{z}_k of z_k whose whole orbit is in \tilde{V}^c when k is large enough. However, such lifts are finite (at most $K_{\tilde{a}, \tilde{b}}$). This implies that there exists a sequence $\{\tilde{z}_k\}_{k \geq 1}$ with $\tilde{z}_k \in \pi^{-1}(z_k)$ such that $\lim_{k \rightarrow +\infty} \rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}_k) = +\infty$, which conflicts with Lemma 1.9. \square

In Example 8.2 of Appendix, we will construct an identity isotopy I of a closed surface such that I satisfies the B-property but its time-one map is not a diffeomorphism and there are two different fixed points \tilde{z}_0 and \tilde{z}_1 of \tilde{F} such that the linking number $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z)$ is not uniformly bounded for $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{z}_0, \tilde{z}_1\})$.

In order to study the boundedness and continuity of the generalize action in the next section, we need the following proposition:

Proposition 5.9. *Let $F \in \text{Diff}^1(M)$ be the time-one map of I and $\tilde{P} \subset \tilde{M}$ be a connected compact set. There exists $N_{\tilde{P}} \geq 0$ such that, for every two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} in \tilde{P} , and $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, we have $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq N_{\tilde{P}}$ when $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists.*

Proof. We make a proof by contradiction. If it is not true, without loss of generality, we suppose that there is a sequence $\{(\tilde{a}_k, \tilde{b}_k)\}_{k \geq 1} \subset \text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F}) \setminus \tilde{\Delta}$ and a sequence $\{z_k\}_{k \geq 1}$ where $z_k \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}_k, \tilde{b}_k\})$ such that $\lim_{k \rightarrow +\infty} i(\tilde{F}; \tilde{a}_k, \tilde{b}_k, z_k) = +\infty$. As \tilde{P} is compact, by extracting subsequences, we can suppose that there exist two fixed points \tilde{a} and \tilde{b} of \tilde{F} in \tilde{P} such that $\lim_{k \rightarrow +\infty} \tilde{a}_k = \tilde{a}$ and $\lim_{k \rightarrow +\infty} \tilde{b}_k = \tilde{b}$.

We identify the sphere \mathbf{S} as the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Recall that $\tilde{I} = (\tilde{F}_t)_{t \in [0, 1]}$ is the lifted identity isotopy of I on the universal cover \tilde{M} of M . Replacing v_i, w_i ($i = 1, 2, 3$) by $v_1(k, t) = \tilde{F}_t(\tilde{a}_k), v_2(k, t) = \tilde{F}_t(\tilde{b}_k), w_1(k, t) = \tilde{a}_k, w_2(k, t) = \tilde{b}_k, v_3(k, t) = w_3(k, t) = \infty$ ($t \in [0, 1]$) in the matrices in the proof of Lemma 1.2, we get the matrix functions $a_t(k), b_t(k), c_t(k)$ and $d_t(k)$.

Let

$$\mathcal{M}_k(t, z) = \frac{a_t(k)z + b_t(k)}{c_t(k)z + d_t(k)}$$

and

$$\tilde{I}_k(z)(t) = \mathcal{M}_k(t, F_t(z)).$$

By the construction, \tilde{I}_k is an isotopy on \tilde{M} from $\text{Id}_{\tilde{M}}$ to \tilde{F} that fixes \tilde{a}_k and \tilde{b}_k .

Similarly, we can construct

$$\mathcal{M}'_k(t, z) = \frac{a'_t(k)z + b'_t(k)}{c'_t(k)z + d'_t(k)}, \quad \mathcal{M}''_k(t, z) = \frac{a''_t(k)z + b''_t(k)}{c''_t(k)z + d''_t(k)}$$

and

$$\tilde{I}'_k(z)(t) = \mathcal{M}'_k(t, F_t(z)), \quad \tilde{I}''_k(z)(t) = \mathcal{M}''_k(t, F_t(z))$$

such that \tilde{I}'_k (resp. \tilde{I}''_k) is an isotopy on \widetilde{M} from $\text{Id}_{\widetilde{M}}$ to \tilde{F} that fixes \tilde{a} (resp. \tilde{b}) and \tilde{a}_k (resp. \tilde{b}_k). As \tilde{I}_k (resp. $\tilde{I}'_k, \tilde{I}''_k$) fixes ∞ , we have $c_t(k) = 0$ (resp. $c'_t(k) = 0, c''_t(k) = 0$) and $a_t(k) d_t(k) \neq 0$ (resp. $a'_t(k) d'_t(k) \neq 0, a''_t(k) d''_t(k) \neq 0$) for all $t \in [0, 1]$ and $k \geq 1$.

Fix an open disk $\widetilde{W} = \{z \in \mathbb{C} \mid |z| > p\}$ that contains ∞ and is disjoint from \tilde{P} . As $\lim_{k \rightarrow +\infty} \tilde{a}_k = \tilde{a}$ $\lim_{k \rightarrow +\infty} \tilde{b}_k = \tilde{b}$ and all of the matrix functions above are continuous on $\tilde{P} \times [0, 1]$, it is easy to see that the norms of these functions have a maximal value $p_{\max} > 0$ and a minimal value $p_{\min} > 0$. Define the open disk

$$\tilde{V} = \left\{ z \in \mathbb{C} \mid |z| > \frac{(p+1)p_{\max}}{p_{\min}} \right\}.$$

Obviously, $\tilde{V} \subset \widetilde{W}$ containing ∞ and for every $\tilde{z} \in \tilde{V}$, we have $\tilde{I}_k(\tilde{z}) \subset \widetilde{W}$, $\tilde{I}'_k(\tilde{z}) \subset \widetilde{W}$ and $\tilde{I}''_k(\tilde{z}) \subset \widetilde{W}$ for all $k \geq 1$. Let \tilde{a}' and \tilde{b}' be two distinct fixed points of \tilde{F} in \tilde{P} . As the linking number $i(\tilde{F}; \tilde{a}', \tilde{b}', z)$ does not depend on the choice of $\tilde{\gamma}$ that joins \tilde{a}' and \tilde{b}' (see 4.1), we can suppose $\tilde{\gamma} \subset \tilde{P}$ in this proof when we talk of the linking number $i(\tilde{F}; \tilde{a}', \tilde{b}', z)$. For \widetilde{W} and \tilde{V} here, Lemma 5.6 and Lemma 5.7 are still valid.

As $F \in \text{Diff}^1(M)$, by Proposition 1.11, I satisfies the B-property. Consider the annulus $A_{\tilde{a}_k, \tilde{b}_k}$. Similarly to the proof of Proposition 5.8, we have $\tilde{z}_k \in \text{Rec}^+(\tilde{F}) \setminus \text{Fix}(\tilde{F})$ where $\tilde{z}_k \in \pi^{-1}(z_k)$ when k is large enough.

For every k , we choose an open disk U_k containing z_k . Let $\Phi_k(z)$ be the first return map of $z \in U_k$ and $\tau(k, z)$ be the first return time of z in this proof. Recall that $\tau_n(k, z) = \sum_{i=0}^{n-1} \tau(k, \Phi_k^i(z))$.

In the proof of Proposition 4.6, we have proved that, for every k and any subsequence $\{\Phi_k^{n_l}(z_k)\}_{l \geq 1}$ which converges to z_k , we have

$$(5.1.6) \quad \frac{L_{n_l}(\tilde{F}; \tilde{a}, \tilde{a}_k, z_k) + L_{n_l}(\tilde{F}; \tilde{a}_k, \tilde{b}_k, z_k) + L_{n_l}(\tilde{F}; \tilde{b}_k, \tilde{b}, z_k)}{\tau_{n_l}(k, z_k)} = 0.$$

On one hand, by the definition of linking number, we have

$$i(\tilde{F}; \tilde{a}_k, \tilde{b}_k, z_k) = \lim_{l \rightarrow +\infty} \frac{L_{n_l}(\tilde{F}; \tilde{a}_k, \tilde{b}_k, z_k)}{\tau_{n_l}(k, z_k)}.$$

As $\lim_{k \rightarrow +\infty} i(\tilde{F}; \tilde{a}_k, \tilde{b}_k, z_k) = +\infty$, we have that, for any $N > 0$, there is $K_N \in \mathbb{N}$ such that when $l, k \geq K_N$,

$$(5.1.7) \quad \frac{L_{n_l}(\tilde{F}; \tilde{a}_k, \tilde{b}_k, z_k)}{\tau_{n_l}(k, z_k)} > N.$$

On the other hand, let us study $\frac{L_{n_l}(\tilde{F}; \tilde{a}, \tilde{a}_k, z_k)}{\tau_{n_l}(k, z_k)}$ and $\frac{L_{n_l}(\tilde{F}; \tilde{b}_k, \tilde{b}, z_k)}{\tau_{n_l}(k, z_k)}$ when l and k are large enough. By the proof of Lemma 5.6 and the conclusion (2) of Lemma 5.7, we only need consider the lift \tilde{z}_k of z_k whose whole orbit is in \tilde{V}^c when k is large enough. Note that

such lifts are finite. Observing the proof of Proposition 1.11, there exists $N' \geq 0$ such that

$$\left| \frac{L_{n_l}(\tilde{F}; \tilde{a}, \tilde{a}_k, z_k)}{\tau_{n_l}(k, z_k)} \right| \leq N' \quad \text{and} \quad \left| \frac{L_{n_l}(\tilde{F}; \tilde{b}_k, \tilde{b}, z_k)}{\tau_{n_l}(k, z_k)} \right| \leq N'$$

when k and l are large enough, which conflicts with Equation 5.1.6 and Inequation 5.1.7. We have completed the proof. \square

In Example 8.3 of Appendix, we will construct an identity isotopy I whose time-one map is a diffeomorphism but not a C^1 -diffeomorphism, that does not satisfy the conclusion of Proposition 5.9.

5.2. Existence and Boundedness in the conservative case.

Proposition 5.10. *Suppose that I satisfies the WB-property at \tilde{a} and \tilde{b} . If $\mu \in \mathcal{M}(F)$ satisfies $\mu(\pi(\tilde{a})) = \mu(\pi(\tilde{b})) = 0$, then μ -almost every point $z \in \text{Rec}^+(F)$ has a rotation vector $\rho_{M,I}(z) \in H_1(M, \mathbb{R})$ and has a linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z) \in \mathbb{R}$. There exists $C > 0$ such that, for every point z such that $\rho_{M,I}(z)$ exists and is not equal to zero, one has $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq C$ if this linking number exists.*

Proof. According to Poincaré Recurrence Theorem, we have $\mu(\text{Rec}^+(F)) = \mu(M)$.

When $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, by Section 1.3.2 and 5.1, $\rho_{M,I}(z)$ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exist and are bounded. Thus we only need to consider the non-fixed point case.

Fix a free open disk $U \subset M \setminus \pi(\{\tilde{a}, \tilde{b}\})$ with $\mu(U) > 0$. For any $z \in \text{Rec}^+(F) \cap U$, by Lemma 5.4 and Lemma 5.6, we have $|L_1(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq \tau(z)K_{\tilde{a},\tilde{b}}(N_{\tilde{a},\tilde{b}} + 1)$ if $\alpha_{z,1} \neq e$ and $|L_1(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq \tau(z)K_{\tilde{a},\tilde{b}}K_U$ if $\alpha_{z,1} = e$. This implies that $L_1(\tilde{F}; \tilde{a}, \tilde{b}, z) \in L^1(U, \mathbb{R}, \mu)$. By Birkhoff Ergodic Theorem, we deduce that the sequence $\{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)/n\}_{n=1}^{+\infty}$ converges to a real number $L^*(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for μ -almost every point on $\text{Rec}^+(F) \cap U$. Recall that, for μ -almost every point on $\text{Rec}^+(F) \cap U$, the sequence $\{\tau_n(z)/n\}_{n=1}^{+\infty}$ converges to a real number $\tau^*(z)$ (see 1.3.2).

We can define the linking number on U as follows (modulo sets of measure zero):

$$(5.2.1) \quad i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau_n(z)} = \frac{L^*(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau^*(z)}.$$

By Proposition 5.1, the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ has a bound K_U for μ -almost every point $z \in \text{Rec}^+(F) \cap U$. As U is arbitrarily chosen, this implies that we can define the function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for μ -almost every point $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}\})$.

Finally, by Proposition 5.5, we can uniformly bound $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ if $\rho_{M,I}(z) \neq 0$. \square

Remark here that, under the hypothesis of Proposition 5.10, $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is bounded on U , but does not necessarily possess a uniform bound on $M \setminus \pi(\{\tilde{a}, \tilde{b}\})$ (see Example 8.2 in Appendix). However, when F is a diffeomorphism of M (see Proposition 5.8), we can get a uniform bound. Moreover, we can get a uniform bound in the case where the support of the measure is the whole space, as stated in the following proposition.

Proposition 5.11. *With the same hypotheses as Proposition 5.10 and if furthermore $\mu \in \mathcal{M}(F)$ has total support, we have $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$ if it exists.*

Proof. The measure μ may naturally be lifted to a (non finite) measure $\tilde{\mu}$ on \tilde{M} . Since μ does not charge $\pi(\tilde{a})$ and $\pi(\tilde{b})$, $\tilde{\mu}$ can be seen as a measure on $A_{\tilde{a}, \tilde{b}}$ invariant by $\tilde{F}_{\tilde{a}, \tilde{b}}$ satisfying $\tilde{\mu}(A_{\tilde{a}, \tilde{b}}) = +\infty$. As the support of $\tilde{\mu}$ is \tilde{M} and $\tilde{F}_{\tilde{a}, \tilde{b}}$ preserves the measure $\tilde{\mu}$, the homeomorphism $\tilde{F}_{\tilde{a}, \tilde{b}}$ satisfies the intersection property, that is, any simple closed curve of $A_{\tilde{a}, \tilde{b}}$ which is not null-homotopic meets its image by $\tilde{F}_{\tilde{a}, \tilde{b}}$. Indeed, any closed curve which goes through ∞ will meet its image by $\tilde{F}_{\tilde{a}, \tilde{b}}$ since $\tilde{F}_{\tilde{a}, \tilde{b}}$ fixes the point ∞ . If the closed curve does not pass through ∞ , we may go back to \tilde{M} and consider a component enclosed by the closed curve which contains \tilde{a} or \tilde{b} and which has finite measure, then it will meet its image since \tilde{F} preserves the measure $\tilde{\mu}$.

In the case where $z \in \text{Fix}(F)$, it is obvious that $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is uniformly bounded.

Choose any free open disk $U \subset M \setminus \text{Fix}(F)$, according to Lemma 5.4, we only need to consider the points $z \in \text{Rec}^+(F) \cap U$ such that $\alpha_{z, n} = e$ for n large enough. We suppose that z is a such point and $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists. We go to the annulus $A_{\tilde{a}, \tilde{b}}$, for any lift \tilde{z} of

z , then we have $\rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}) = \lim_{n \rightarrow +\infty} \frac{\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n}{\tau_n(z)}$.

We claim that, for any $\epsilon > 0$, $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq (N_{\tilde{a}, \tilde{b}} + 1 + \epsilon)K_{\tilde{a}, \tilde{b}}$. Otherwise, without loss of generality, we can suppose that $i(\tilde{F}; \tilde{a}, \tilde{b}, z) > (N_{\tilde{a}, \tilde{b}} + 1 + \epsilon)K_{\tilde{a}, \tilde{b}}$. Then there exists a number N large enough such that for every $n \geq N$, there is a lift \tilde{z}_n of z in \tilde{V}^c satisfying $\frac{\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}_n}^n}{\tau_n(z)} > N_{\tilde{a}, \tilde{b}} + 1 + \epsilon$. This implies that there exists a lift \tilde{z} of z in \tilde{V}^c such that $\rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}) \geq N_{\tilde{a}, \tilde{b}} + 1 + \epsilon > N_{\tilde{a}, \tilde{b}} + 1$. By the fact $\rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\infty) = 0$ and according to Theorem 3.5, $\tilde{F}_{\tilde{a}, \tilde{b}}$ has a fixed point whose rotation number is $N_{\tilde{a}, \tilde{b}} + 1$, which is a contradiction. This proves the claim.

As ϵ is arbitrarily chosen, we get $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$. \square

The function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is not necessarily μ -integrable (see Example 8.2 in Appendix). But in some cases, as we have stated above, where the time-one map F is a diffeomorphism of M , or I satisfies the WB-property at \tilde{a} and \tilde{b} , and μ is ergodic (because it is constant μ -a.e.) or the support of μ is the whole space, the function $i_\mu(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is μ -integrable.

Suppose now the function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is μ -integrable. We can define a function as follows

$$(5.2.2) \quad i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = \int_{M \setminus \pi(\{\tilde{a}, \tilde{b}\})} i(\tilde{F}; \tilde{a}, \tilde{b}, z) d\mu.$$

From Propositions 4.3 and 4.4, we get the following corollaries immediately:

Corollary 5.12. *We have $i_\mu(\tilde{F}^q; \tilde{a}, \tilde{b}) = q i_\mu(\tilde{F}; \tilde{a}, \tilde{b})$ for all $q \geq 1$.*

Corollary 5.13. *We have $i_\mu(\tilde{F}; \alpha(\tilde{a}), \alpha(\tilde{b})) = i_\mu(\tilde{F}; \tilde{a}, \tilde{b})$ for any $\alpha \in G$.*

Let H be an orientation preserving homeomorphism of M and \tilde{H} be a lift of H to \tilde{M} . From Proposition 4.5, we get the following corollary.

Corollary 5.14. *We have $i_{H_*(\mu)}(\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}; \tilde{H}(\tilde{a}), \tilde{H}(\tilde{b})) = i_\mu(\tilde{F}; \tilde{a}, \tilde{b})$.*

In the end of this section, we will give the integral 5.2.2 a geometric description when F and F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$. Before that, let us introduce a definition.

Let $\mathbb{A} = \mathbf{T}^1 \times [0, 1]$ be a closed annulus and let T be the generator of the covering transformation group $\pi : \tilde{\mathbb{A}} \rightarrow \mathbb{A}$ where $\tilde{\mathbb{A}} = \mathbb{R} \times [0, 1]$. Suppose that $J = (h_t)_{t \in [0, 1]}$ is an isotopy of \mathbb{A} from $\text{Id}_{\mathbb{A}}$ to h , ν is a Borel measure (ν is admitted to be an infinite measure here) invariant by h on \mathbb{A} . Let $\gamma : [0, 1] \rightarrow \mathbb{A}$ be a simple oriented path which satisfies $\gamma(0) \in \mathbf{T}^1 \times \{0\}$, $\gamma(1) \in \mathbf{T}^1 \times \{1\}$ and $\text{Int}(\gamma) \subset \text{Int}(\mathbb{A})$. Denote by $\Sigma' : [0, 1] \times [0, 1] \rightarrow \mathbb{A}$ the 2-chain $\Sigma'(s, t) = h_s^{-1}(\gamma(t))$ and by $|\Sigma'| = \{z \in \mathbb{A} \mid z = h_s^{-1}(\gamma(t)), (s, t) \in [0, 1] \times [0, 1]\}$ the support of Σ' . When $\nu(\gamma) = 0$, the intersection number $\gamma \wedge J(z)$ is well defined for ν -almost every z on \mathbb{A} . Define the algebraic area of the 2-chain Σ' in \mathbb{A} , that is, the algebraic area (for ν) “swept out” by $\bigcup_{s \in [0, 1]} h_s^{-1}(\gamma)$, as follows

$$\int_{\Sigma'} d\nu = \int_{\mathbb{A}} \gamma \wedge J(z) d\nu.$$

When $\nu(|\Sigma'|) < +\infty$, the integral is well defined. Indeed, there exist a number $N \geq 0$ such that $|\gamma \wedge J(z)| \leq N$ since \mathbb{A} is compact. Obviously, $\gamma \cap J(z) = \emptyset$ if $z \notin \bigcup_{s \in [0, 1]} h_s^{-1}(\gamma(t))$. Therefore,

$$\left| \int_{\Sigma'} d\nu \right| \leq \int_{\mathbb{A}} |\gamma \wedge J(z)| d\nu \leq \nu(|\Sigma'|)N < +\infty.$$

Let H be the lift of h that is the time-one map of the lifted identity isotopy \tilde{J} of J , $\tilde{\gamma}$ be a connected component of γ in $\tilde{\mathbb{A}}$ and $\tilde{\nu}$ be the lift of ν to $\tilde{\mathbb{A}}$. Let \tilde{D}' be the closed region between $H^{-1}(\tilde{\gamma})$ and $T(H^{-1}(\tilde{\gamma}))$ which is a fundamental domain of T . We have

$$(5.2.3) \quad \int_{\Sigma'} d\nu = \int_{\mathbb{A}} \gamma \wedge J(z) d\nu = \int_{\tilde{D}'} \tilde{\gamma} \wedge \tilde{J}(\tilde{z}) d\tilde{\nu}$$

which does not depend on the choice of $\tilde{\gamma}$.

Denote by $\Sigma = h * \Sigma' : [0, 1] \times [0, 1] \rightarrow \mathbb{A}$ the 2-chain $\Sigma(s, t) = h_s^{-1}(h(\gamma(t)))$ and suppose that $\nu(|\Sigma|) < +\infty$. Let $\tilde{D} = H(\tilde{D}')$ be the closed region between $\tilde{\gamma}$ and $T(\tilde{\gamma})$ which is also a fundamental domain of T . By Equation 5.2.3, we have

$$(5.2.4) \quad \int_{\Sigma} d\nu = \int_{\mathbb{A}} h(\gamma) \wedge J(z) d\nu = \int_{\tilde{D}} H(\tilde{\gamma}) \wedge \tilde{J}(\tilde{z}) d\tilde{\nu}.$$

The equation 5.2.4 tell us that the value $\int_{\Sigma} d\nu$ is equal to the algebraic area (for $\tilde{\nu}$) of the region of $\tilde{\mathbb{A}}$ situated between $\tilde{\gamma}$ and its image $H(\tilde{\gamma})$. Furthermore, if we suppose that J

fixes a point ∞ in \mathbb{A} , we have

$$\begin{aligned}
 (5.2.5) \quad \int_{\Sigma} d\nu &= \int_{\mathbb{A}} h(\gamma) \wedge J(z) d\nu \\
 &= \int_{\mathbb{A}} \gamma \wedge (h^{-1} \circ J)(z) d\nu \\
 &= \int_{\mathbb{A}} \gamma \wedge (h^{-1} \circ J \circ h)(z) d\nu \\
 &= \int_{\mathbb{A}} \gamma \wedge J(z) d\nu.
 \end{aligned}$$

Indeed, write the isotopy $J' = h^{-1} \circ J \circ h = (h^{-1} \circ h_t \circ h)_{0 \leq t \leq 1}$. The third equation holds because h is a homeomorphism of \mathbb{A} and preserves the measure ν . Observing that the isotopy $J^{-1}J'$ is a loop (whose base point is $\text{Id}_{\mathbb{A}}$) in $\text{Homeo}_*(\mathbb{A})$ and fixes the point ∞ , recall that $\pi_1(\text{Homeo}_*(\mathbb{A})) = \bigcup_{k \in \mathbb{Z}} \mathcal{C}_k$ (see the proof of Proposition 4.6), we get $[J^{-1}J']_1 \in \mathcal{C}_0$. Hence, we get the last equation. It is easy to prove that, by induction and Equation 5.2.5, $\int_{\Sigma} d\nu$ is equal to $\int_{h^k \Sigma'} d\nu$ for every $k \in \mathbb{Z}$.

Remark that we can also define the algebraic area of the 2-chain Σ when γ is not simple if we consider the oriented domain enclosed by $\tilde{\gamma}$, $H(\tilde{\gamma})$ and $\partial \tilde{\mathbb{A}}$ in $\tilde{\mathbb{A}}$. However, to prove Theorem 0.1 in the next section, it is enough to merely consider the case of a simple oriented path.

Suppose now the measure ν is defined by a symplectic form ω , that is, $\nu(A) = \int_A \omega$ for all measurable sets $A \subset \mathbb{A}$. Observe that $\tilde{\omega}$ is exact in $\tilde{\mathbb{A}}$ where $\tilde{\omega}$ is the lift of ω to $\tilde{\mathbb{A}}$. The equation 5.2.4 and Stokes' theorem imply that $\int_{\Sigma} \omega$ (defined by the integrals of differential 2-form on 2-chain) is nothing else but the algebraic area of the 2-chain Σ in \mathbb{A} , $\int_{\Sigma} d\nu$ (defined by Equation 5.2.4).

We now suppose that the time-one map F of I and its inverse F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$. Let $\tilde{I}_1 = (\tilde{F}'_t)_{t \in [0,1]}$ be an isotopy from $\text{Id}_{\tilde{M}}$ to \tilde{F} that fixes \tilde{a} and \tilde{b} , and $\tilde{\mu}$ be the lift of μ to \tilde{M} . Let $\tilde{\gamma} : [0,1] \rightarrow \tilde{M}$ be a simple oriented path from \tilde{a} to \tilde{b} with $\tilde{\gamma}(0) = \tilde{a}$ and $\tilde{\gamma}(1) = \tilde{b}$. Consider the annulus $A_{\tilde{a},\tilde{b}}$ and the annulus map $\tilde{F}_{\tilde{a},\tilde{b}}$. Recall that, in the proof of Lemma 1.9, $\bar{A}_{\tilde{a},\tilde{b}} = S_{\tilde{a}} \sqcup A_{\tilde{a},\tilde{b}} \sqcup S_{\tilde{b}}$ is the natural compactification of $A_{\tilde{a},\tilde{b}}$ where $S_{\tilde{a}}$ and $S_{\tilde{b}}$ are the tangent unit circles at \tilde{a} and \tilde{b} . We can identify $\tilde{\gamma}$ as an oriented path in $\bar{A}_{\tilde{a},\tilde{b}}$ and \tilde{I}_1 as an identity isotopy of $\bar{A}_{\tilde{a},\tilde{b}}$. As the measure $\tilde{\mu}$ is invariant by \tilde{F} and $\tilde{\mu}(\tilde{a}) = \tilde{\mu}(\tilde{b}) = 0$, it naturally induces a measure on $\bar{A}_{\tilde{a},\tilde{b}}$, denoted still by $\tilde{\mu}$.

Suppose that $\tilde{\Sigma}$ is the 2-chain $\tilde{\Sigma} : [0,1] \times [0,1] \rightarrow \tilde{M}$ defined by $\tilde{\Sigma}(s,t) = \tilde{F}'_s^{-1}(\tilde{F}(\tilde{\gamma}(t)))$ whose boundary is $\tilde{F}(\tilde{\gamma})\tilde{\gamma}^{-1}$ with the boundary of the square $[0,1] \times [0,1]$ oriented counter-clockwise. As \tilde{I}_1 fixes ∞ , the intersection number $\tilde{\gamma} \wedge \tilde{I}_1(\tilde{z})$ is zero when \tilde{z} belongs to a neighborhood of ∞ . Therefore, if $\tilde{\mu}(\tilde{\gamma}) = 0$, we can define the algebraic area of the 2-chain $\tilde{\Sigma}$ in $\tilde{M} \setminus \{\tilde{a}, \tilde{b}\}$ as follows

$$\int_{\tilde{\Sigma}} d\tilde{\mu} = \int_{\tilde{M} \setminus \{\tilde{a}, \tilde{b}\}} \tilde{\gamma} \wedge \tilde{I}_1(\tilde{z}) d\tilde{\mu} = \int_{\bar{A}_{\tilde{a},\tilde{b}}} \tilde{\gamma} \wedge \tilde{I}_1(\tilde{z}) d\tilde{\mu}.$$

Remark here that if the measure μ is defined by a symplectic form ω , then $\int_{\widetilde{\Sigma}} \widetilde{\omega}$ (see Equation 2.1.2 and Equation 2.1.4) is nothing else but $\int_{\widetilde{\Sigma}} d\widetilde{\mu}$ where $\widetilde{\omega}$ is the lift of ω to \widetilde{M} . Moreover, we have the following result which is a key step to prove that our generalized action function defined in next section is a generalization of the classical one.

Lemma 5.15. *If $\widetilde{\mu}(\widetilde{\gamma}) = 0$, then we have*

$$i_{\mu}(\widetilde{F}, \widetilde{a}, \widetilde{b}) = \int_{\widetilde{\Sigma}} d\widetilde{\mu}.$$

Proof. From Proposition 5.8, we know that $i_{\mu}(\widetilde{F}, \widetilde{a}, \widetilde{b})$ is well defined. Let

$$Z = \bigcup_{k=0}^{+\infty} (F^{-k}(\pi(\widetilde{\gamma}))).$$

Observe that $\mu(\text{Rec}^+(F) \setminus Z) = \mu(M)$. For every $z \in \text{Rec}^+(F) \setminus Z$ and every $n \geq 1$, consider the following infinite family of paths in \widetilde{M} :

$$\widetilde{\Gamma}_{I_1, z}^n = \prod_{\pi(\widetilde{z})=z} \widetilde{I}_1^n(\widetilde{z}).$$

Define the function

$$G_n(\widetilde{F}; \widetilde{a}, \widetilde{b}, z) = \widetilde{\gamma} \wedge \widetilde{\Gamma}_{I_1, z}^n.$$

Let us verify that this is well defined. Consider the annulus $A_{\widetilde{a}, \widetilde{b}}$ and the annulus map $\widetilde{F}_{\widetilde{a}, \widetilde{b}}$. For any $z \in \text{Rec}^+(F) \setminus Z$, let \widetilde{z} be any lift of z to \widetilde{M} (we also write \widetilde{z} in $A_{\widetilde{a}, \widetilde{b}}$), and \widehat{z} be any lift of \widetilde{z} to $\widehat{A}_{\widetilde{a}, \widetilde{b}}$. In the proof of Lemma 1.9, we have proved that $|p_1(\widehat{F}_{\widetilde{a}, \widetilde{b}}(\widehat{z})) - p_1(\widehat{z})|$ is uniformly bounded for any $\widehat{z} \in \widehat{A}_{\widetilde{a}, \widetilde{b}}$, say N as a bound, and depends on the isotopy I but not on the choice of \widehat{z} . Fix an open disk \widetilde{W} containing ∞ and disjoint from $\widetilde{\gamma}$. As $\widetilde{I}_1(\infty) = \infty$, for every $n \geq 1$, we can choose an open disk $\widetilde{V}_n \subset \widetilde{W}$ containing ∞ such that for every $\widetilde{z} \in \widetilde{V}_n$, we have $\widetilde{I}_1^n(\widetilde{z}) \in \widetilde{W}$. Write $X_z^m = \pi^{-1}(\{z\}) \cap \widetilde{V}_n^c$. We deduce that there is a positive integer K'_n such that $\sharp X_z^m \leq K'_n$ and

$$\left| G_n(\widetilde{F}; \widetilde{a}, \widetilde{b}, z) \right| = \left| \widetilde{\gamma} \wedge \widetilde{\Gamma}_{I_1, z}^n \right| = \left| \sum_{\widetilde{z} \in X_z^m} \widetilde{\gamma} \wedge \widetilde{I}_1^n(\widetilde{z}) \right| \leq K'_n N.$$

Hence we complete the claim. As a consequence, $G_1(\widetilde{F}; \widetilde{a}, \widetilde{b}, z) \in L^1(M \setminus \pi(\{\widetilde{a}, \widetilde{b}\}), \mathbb{R}, \mu)$.

Moreover, we can write $G_n(\widetilde{F}; \widetilde{a}, \widetilde{b}, z)$ as a Birkhoff sum:

$$G_n(\widetilde{F}; \widetilde{a}, \widetilde{b}, z) = \widetilde{\gamma} \wedge \widetilde{\Gamma}_{I_1, z}^n = \widetilde{\gamma} \wedge \prod_{i=0}^{n-1} \widetilde{\Gamma}_{I_1, F^i(z)}^1 = \sum_{j=0}^{n-1} G_1(\widetilde{F}; \widetilde{a}, \widetilde{b}, F^j(z)).$$

According to Birkhoff Ergodic theorem, the limit

$$\lim_{n \rightarrow +\infty} \frac{G_n(\widetilde{F}; \widetilde{a}, \widetilde{b}, z)}{n} = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} G_1(\widetilde{F}; \widetilde{a}, \widetilde{b}, F^j(z))$$

exists for μ -almost everywhere on $M \setminus \pi(\{\tilde{a}, \tilde{b}\})$. We know that

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau_n(z)} = \frac{L^*(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau^*(z)}$$

for μ -almost every point $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}\})$ exists (see Proposition 5.10). As $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ does not depend on the choice of U (see Definition 4.1), when $z \notin \pi(\tilde{\gamma})$, we can suppose that the disk U is small enough such that $U \cap \pi(\tilde{\gamma}) = \emptyset$. Therefore, $\{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)/\tau_n(z)\}_{n \geq 1}$ is a subsequence of $\{G_n(\tilde{F}; \tilde{a}, \tilde{b}, z)/n\}_{n \geq 1}$. We get

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \lim_{n \rightarrow +\infty} \frac{G_n(\tilde{F}; \tilde{a}, \tilde{b}, z)}{n}$$

for μ -almost everywhere on $M \setminus \pi(\{\tilde{a}, \tilde{b}\})$.

By Birkhoff Ergodic theorem, we have

$$\begin{aligned} i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) &= \int_{M \setminus \pi(\{\tilde{a}, \tilde{b}\})} i(\tilde{F}; \tilde{a}, \tilde{b}, z) d\mu \\ &= \int_{M \setminus \pi(\{\tilde{a}, \tilde{b}\})} G_1(\tilde{F}; \tilde{a}, \tilde{b}, z) d\mu \\ &= \int_{\tilde{M} \setminus \pi^{-1}(\pi(\{\tilde{a}, \tilde{b}\}))} \tilde{\gamma} \wedge \tilde{I}_1(\tilde{z}) d\tilde{\mu} \\ &= \int_{\tilde{\Sigma}} d\tilde{\mu}, \end{aligned}$$

We have completed the proof. \square

6. ACTION FUNCTION

This section will be divided into three parts. In the first part, we will define the action function and prove Theorem 0.1. In the second part, we will study some properties of the action. In the third part, we will define the action spectrum and prove that the action is not constant in the case where the contractible fixed points set is finite.

Firstly, we state some results we can already get immediately. Let $F \in \text{Homeo}_*(M)$ be the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ of M . As we have proved in the last section, we know that the function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is μ -integrable for every pair $(\tilde{a}, \tilde{b}) \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ in each of the following cases:

- $F \in \text{Diff}(M)$, and $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont}, I}(F)$ (Proposition 5.8);
- I satisfies the WB-property, and $\mu \in \mathcal{M}(F)$ has total support but no atoms on $\text{Fix}_{\text{Cont}, I}(F)$ (Proposition 5.11);
- I satisfies the WB-property, $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont}, I}(F)$ and μ is ergodic (Proposition 5.10 and the Birkhoff Ergodic theorem).

6.1. Definition of the action function. In this subsection, we suppose that the function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is μ -integrable for every two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} .

We define the *action difference* as follows:

$$\begin{aligned} i_\mu : (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta} &\rightarrow \mathbb{R} \\ (\tilde{a}, \tilde{b}) &\mapsto i_\mu(\tilde{F}; \tilde{a}, \tilde{b}). \end{aligned}$$

From Proposition 4.6, we have the following corollary immediately:

Corollary 6.1. *For any distinct fixed points \tilde{a} , \tilde{b} and \tilde{c} of \tilde{F} , we have*

$$i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) + i_\mu(\tilde{F}; \tilde{b}, \tilde{c}) + i_\mu(\tilde{F}; \tilde{c}, \tilde{a}) = 0.$$

That is, i_μ is a coboundary on $\text{Fix}(\tilde{F})$. So there is a function $l_\mu : \text{Fix}(\tilde{F}) \rightarrow \mathbb{R}$, defined up to an additive constant, such that

$$(6.1.1) \quad i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = l_\mu(\tilde{F}; \tilde{b}) - l_\mu(\tilde{F}; \tilde{a}).$$

We call the function l_μ the *action* on $\text{Fix}(\tilde{F})$ defined by the measure μ .

As a consequence, if F is a diffeomorphism of M and the measure $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont}, I}(F)$, or the isotopy I satisfies the WB-property and the measure $\mu \in \mathcal{M}(F)$ has total support but no atoms on $\text{Fix}_{\text{Cont}, I}(F)$, then the action function is well defined on $\text{Fix}(\tilde{F})$, but the action can be unbounded. In Example 8.3 of Appendix, we will construct an isotopy I and a measure $\mu \in \mathcal{M}(F)$ such that the time-one map F is a diffeomorphism (hence I satisfies the WB-property), and the measure μ has total support but no atoms on $\text{Fix}_{\text{Cont}, I}(F)$, while the action is unbounded.

Proposition 6.2. *If $\rho_{M, I}(\mu) = 0$, then $i_\mu(\tilde{F}; \tilde{a}, \alpha(\tilde{a})) = 0$ for every $\tilde{a} \in \text{Fix}(\tilde{F})$ and every $\alpha \in G^*$. As a consequence, there exists a function L_μ defined on $\text{Fix}_{\text{Cont}, I}(F)$ such that for every two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , we have*

$$i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = L_\mu(\tilde{F}; \pi(\tilde{b})) - L_\mu(\tilde{F}; \pi(\tilde{a})).$$

Proof. There exists an isotopy I' homotopic to I that fixes $\pi(\tilde{a})$. It is lifted to an isotopy \tilde{I}' that fixes \tilde{a} and $\alpha(\tilde{a})$. Observe that if $\tilde{\gamma}$ is an oriented path from \tilde{a} to $\alpha(\tilde{a})$, then the intersection number $\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}', z}^n$ (see 4.1) is equal to the intersection between the loop $\pi(\tilde{\gamma})$ and the loop $I'^{\tau_n(z)}(z) \gamma_{\Phi^n(z), z}$ (see 1.3.2). As $\rho_{M, I}(\mu) = \rho_{M, I'}(\mu) = 0$ and $\pi(\tilde{a}) \in \text{Fix}_{\text{Cont}, I}(F)$ (or $\mu(\pi(\tilde{a})) = 0$), we have

$$\begin{aligned} i_\mu(\tilde{F}; \tilde{a}, \alpha(\tilde{a})) &= \int_{M \setminus \{\pi(\tilde{a})\}} i(\tilde{F}; \tilde{a}, \alpha(\tilde{a}), z) d\mu \\ &= \int_{M \setminus \{\pi(\tilde{a})\}} \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}, \alpha(\tilde{a}), z)}{\tau_n(z)} d\mu \\ &= \int_{M \setminus \{\pi(\tilde{a})\}} \lim_{n \rightarrow +\infty} \frac{\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}', z}^n}{\tau_n(z)} d\mu \\ &= \pi(\tilde{\gamma}) \wedge \rho_{M, I'}(\mu) \\ &= 0 \end{aligned}$$

The second conclusion follows from Corollary 6.1. We have completed the proof. \square

We call the function L_μ the *action function* or *action* on $\text{Fix}_{\text{Cont}, I}(F)$ defined by the measure μ .

Proof of Theorem 0.1. From Corollary 6.1 and Proposition 6.2, we define the *action difference* $I_\mu : (\text{Fix}_{\text{Cont}, I}(F) \times \text{Fix}_{\text{Cont}, I}(F)) \setminus \Delta \rightarrow \mathbb{R}$ and the action $L_\mu : \text{Fix}_{\text{Cont}, I}(F) \rightarrow \mathbb{R}$ as follows

$$(6.1.2) \quad I_\mu(\tilde{F}; a, b) = i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = L_\mu(\tilde{F}; b) - L_\mu(\tilde{F}; a),$$

where \tilde{a} and \tilde{b} are any lifts of a and b . We only need to prove that the function L_μ defined in this section is a generalization of the action difference in 2.1.2.

Observe that, in the classical case, $I = (F_t)_{t \in [0,1]} \subset \text{Diff}_*(M)$ where $\text{Diff}_*(M)$ is the set of diffeomorphisms that are isotopic to the identity. The measure μ is defined by a symplectic form ω . Therefore, μ is non-atomic. Comparing the Equation 2.1.4 with Equation 6.1.2, it sufficient to prove that $I_\mu(\tilde{F}; a, b) = i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = \delta(\tilde{F}, \tilde{a}, \tilde{b})$.

Let $\tilde{\gamma}$ be any oriented path from \tilde{a} to \tilde{b} . By Lemma 5.15, we have

$$i_\mu(\tilde{F}, \tilde{a}, \tilde{b}) = \int_{\tilde{\Sigma}} d\tilde{\mu}$$

where $\tilde{\Sigma}$ is the 2-chain whose boundary is $\tilde{F}(\tilde{\gamma}) - \tilde{\gamma}$ (that is, identify $\tilde{F}(\tilde{\gamma})\tilde{\gamma}^{-1}$ as a 1-chain) as defined in Lemma 5.15. As $\delta(\tilde{F}, \tilde{a}, \tilde{b})$ does not depend on the choices of $\tilde{\gamma}$ and $\tilde{\Sigma}$ (see 2.1.2), we have

$$i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = \delta(\tilde{F}, \tilde{a}, \tilde{b}).$$

We have completed the proof. \square

6.2. The properties of the action function. In this section, we will give some properties of the action function that we have defined in 6.1.

From Theorem 0.1 and Corollary 5.12, we get the following corollary immediately:

Corollary 6.3. *Under the same hypotheses as Theorem 0.1, for every two distinct contractible fixed points a and b of F , we have $I_\mu(\tilde{F}^q; a, b) = qI_\mu(\tilde{F}; a, b)$ for all $q \geq 1$.*

Let us study the continuity and boundedness of the actions l_μ and L_μ .

Lemma 6.4. *Let $I = (F_t)_{t \in [0,1]}$ be an identity isotopy of \mathbf{S}^2 and $\bar{z}, \bar{\bar{z}} \in \mathbf{S}^2$ be two fixed points of F_1 . If $\{z_n\}_{n \geq 1} \subset \text{Fix}(F_1) \setminus \{\bar{z}, \bar{\bar{z}}\}$ satisfies $z_n \rightarrow \bar{z}$ as $n \rightarrow +\infty$, then for any open neighborhood W of \bar{z} , there exists a positive integer n_W such that for every $n \geq n_W$, there exists an isotopy I_n from $\text{Id}_{\mathbf{S}^2}$ to F_1 that fixes $\bar{z}, \bar{\bar{z}}$ and z_n , and there exists an open neighborhood V_n of \bar{z} containing z_n , such that $I_n(z) \cap V_n = \emptyset$ when $z \notin W$.*

Proof. We identify the sphere \mathbf{S}^2 to the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

For simplicity, up to conjugacy by a Möbius transformation (see the proof of Lemma 1.2) that maps the triple $(\bar{z}, \bar{\bar{z}}, z_1)$ to the triple $(0, 1, z_1)$, we can suppose that $\bar{z} = 0$ and $\bar{\bar{z}} = 1$. We choose an isotopy $I_1 = (F'_t)_{0 \leq t \leq 1}$ fixing the triple $(0, 1, z_1)$ (using Lemma 1.2).

Let

$$\mathcal{M}_n(t, z) = \frac{z_n(F'_t(z_n) - 1)z}{(F'_t(z_n) - z_n)z + F'_t(z_n)(z_n - 1)}$$

and

$$I_n(z)(t) = \mathcal{M}_n(t, F'_t(z)).$$

By construction, I_n is an isotopy from $\text{Id}_{\mathbb{S}^2}$ to F_1 that fixes the triple $(0, 1, z_n)$.

Let W be any open neighborhood of 0 and V_n be the ball whose center is at 0 and radius is $2|z_n|$. Write

$$m(W) = \inf_{(t,z) \in [0,1] \times W^c} |F'_t(z)| \quad \text{and} \quad M(z) = \sup_{t \in [0,1]} |F'_t(z)|.$$

As I_1 fixes 0, we have $m(W) > 0$ and $M(z_n) \rightarrow 0$ as $n \rightarrow +\infty$. Therefore, there exists a positive number n_W such that when $n \geq n_W$,

$$M(z_n) < \min\left\{\frac{1}{2}, \frac{m(W)}{8m(W) + 4}\right\}.$$

For any $z \notin W$, every $n \geq n_W$ and $t \in [0, 1]$, we have

$$\begin{aligned} |I_n(z)(t)| &= \frac{|z_n(F'_t(z_n) - 1)F'_t(z)|}{|(F'_t(z_n) - z_n)F'_t(z) + F'_t(z_n)(z_n - 1)|} \\ &= \frac{|z_n(F'_t(z_n) - 1)|}{|(F'_t(z_n) - z_n) + \frac{F'_t(z_n)(z_n - 1)}{F'_t(z)}|} \\ &\geq \frac{|(F'_t(z_n) - 1)|}{|(F'_t(z_n) - z_n)| + \left|\frac{F'_t(z_n)(z_n - 1)}{m(W)}\right|} |z_n| \\ &> \frac{1/2}{2M(z_n) + M(z_n)/m(W)} |z_n| \\ &> 2|z_n|. \end{aligned}$$

Hence $I_n(z) \cap V_n = \emptyset$. We have completed the proof. \square

Lemma 6.5. *We suppose that $\tilde{a} \in \text{Fix}(\tilde{F}) \setminus \{\infty\}$ and $\{\tilde{a}_n\}_{n \geq 1} \subset \text{Fix}(\tilde{F}) \setminus \{\tilde{a}, \infty\}$ satisfying $\tilde{a}_n \rightarrow \tilde{a}$ as $n \rightarrow +\infty$. Then*

$$\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$$

when $z \in \text{Fix}(F) \setminus \{\pi(\tilde{a})\}$, while

$$\lim_{n \rightarrow +\infty} L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$$

when $z \in \text{Rec}^+(F) \cap U$ where U is a disk of $M \setminus \{\pi(\tilde{a})\}$.

Proof. When $z \in U$, recall that the first return map is $\tau(z)$. For convenience, we write $\tau(z) = 1$ if $z \in \text{Fix}(F)$. For any given $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}\})$, let \tilde{W} be any open neighborhood of \tilde{a} satisfying $\tilde{W} \cap \pi^{-1}(\{z, F(z), \dots, F^{\tau(z)-1}(z)\}) = \emptyset$. By Lemma 6.4, there exist a number $n_{\tilde{W}}$, a family of isotopies $\{\tilde{I}_n\}_{n \geq n_{\tilde{W}}}$ with \tilde{I}_n fixing the points \tilde{a}, ∞

and \tilde{a}_n , and a family of neighborhoods $\{\tilde{V}_n\}_{n \geq n_{\tilde{W}}}$ of \tilde{a} with \tilde{V}_n containing \tilde{a}_n , such that $\tilde{I}_n(\tilde{z}) \cap \tilde{V}_n = \emptyset$ for any $\tilde{z} \in \pi^{-1}(\{z, F(z), \dots, F^{\tau(z)-1}(z)\})$.

The functions $i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)$ when z is a fixed point of F that is disjoint from $\pi(\tilde{a})$ and $\pi(\tilde{a}_n)$, and $L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z)$ when $z \in \text{Rec}^+(F) \cap U$ and n is large enough, depend neither on the choice of the isotopy \tilde{I} that fixes the points \tilde{a} , \tilde{a}_n and ∞ , nor on the path from \tilde{a}_n to \tilde{a} (see 4.1). Therefore, for every $n \geq n_{\tilde{W}}$, we can choose the isotopy \tilde{I}_n as above and a path in \tilde{V}_n from \tilde{a}_n to \tilde{a} . As a consequence, we have

$$\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$$

in the case where $z \in \text{Fix}(F) \setminus \{\pi(\tilde{a})\}$, and

$$\lim_{n \rightarrow +\infty} L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$$

in the case where $z \in \text{Rec}^+(F) \cap U$. □

Lemma 6.6. *Suppose that an isotopy I satisfies the B-property, $\mu \in \mathcal{M}(F)$ is ergodic and no atoms on $\text{Fix}_{\text{Cont}, I}(F)$ where F is the time-one map of I . Let $\tilde{P} \subset \tilde{M}$ be a connected compact set. There exists $N_{\tilde{P}} \geq 0$ such that $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq N_{\tilde{P}}$ for all two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} in \tilde{P} and μ -a.e. z .*

Proof. Take a disk U such that $\mu(U) > 0$. Recall that Φ and τ are respectively the first return map and the first return time. Denote by μ_U the measure on U induced by μ and $\text{Orb}^+(z)$ the positive orbit of z , that is, $\text{Orb}^+(z) = \{F^n(z) | n \geq 0\}$. Then $\mu_U \in \mathcal{M}(\Phi)$ and μ_U is ergodic with regard to Φ . Indeed, if h is a measurable function on U and satisfies $h \circ \Phi = h$. Observe that $\mu(\cup_{n \geq 0} F^n(U)) = \mu(M)$ since μ is ergodic. We may extend h on M in the following way:

$$h'(z') = \begin{cases} h(z) & \text{if } z \in \text{Rec}^+(F) \cap U \text{ and } z' \in \text{Orb}^+(z); \\ 0 & \text{if } z \notin \text{Rec}^+(F) \cap U \text{ and } z' \in \text{Orb}^+(z). \end{cases}$$

By the construction of h' , we have $h' \circ F = h'$. Hence, h' is constant μ -a.e. on M . This implies that h is constant μ_U -a.e. on U and therefore μ_U is ergodic with regard to Φ . Observe that $\int_U \tau(z) d\mu = \mu(\cup_{k \geq 0} F^k(U)) = \mu(M)$. By Birkhoff Ergodic theorem and Equation 5.2.1, for all two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} in \tilde{P} and μ -a.e. z , we have

$$\begin{aligned} i(\tilde{F}; \tilde{a}, \tilde{b}, z) &= \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau_n(z)} = \lim_{n \rightarrow +\infty} \frac{\frac{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)}{n}}{\frac{\tau_n(z)}{n}} \\ &= \frac{\int_U L_1(\tilde{F}; \tilde{a}, \tilde{b}, z) d\mu}{\int_U \tau(z) d\mu} \\ &= \frac{1}{\mu(M)} \int_U L_1(\tilde{F}; \tilde{a}, \tilde{b}, z) d\mu. \end{aligned}$$

If the lemma is not true, then we can find $\{(\tilde{a}_n, \tilde{b}_n)\}_{n \geq 1} \subset \text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F}) \setminus \tilde{\Delta} \cap \tilde{P}$ and $z \in U$ such that $|i(\tilde{F}; \tilde{a}_n, \tilde{b}_n, z)| \geq n$. That is

$$(6.2.1) \quad \lim_{n \rightarrow +\infty} \left| \int_U L_1(\tilde{F}; \tilde{a}_n, \tilde{b}_n, z) d\mu \right| = +\infty.$$

We can suppose that there are two fixed points \tilde{a} and \tilde{b} of \tilde{F} such that $\lim_{n \rightarrow +\infty} \tilde{a}_n = \tilde{a}$ and $\lim_{n \rightarrow +\infty} \tilde{b}_n = \tilde{b}$ by taking subsequences if necessary.

When $\tilde{a} = \tilde{b}$, by Proposition 4.6, we have

$$L_1(\tilde{F}; \tilde{a}_n, \tilde{b}_n, z) + L_1(\tilde{F}; \tilde{b}_n, \tilde{a}, z) + L_1(\tilde{F}; \tilde{a}, \tilde{a}_n, z) = 0.$$

By Lemma 6.5, we have $L_1(\tilde{F}; \tilde{a}_n, \tilde{b}_n, z) \rightarrow 0$ as $n \rightarrow +\infty$.

When $\tilde{a} \neq \tilde{b}$, we have

$$L_1(\tilde{F}; \tilde{a}_n, \tilde{b}_n, z) + L_1(\tilde{F}; \tilde{b}_n, \tilde{b}, z) + L_1(\tilde{F}; \tilde{b}, \tilde{a}, z) + L_1(\tilde{F}; \tilde{a}, \tilde{a}_n, z) = 0.$$

By Lemma 6.5 again, we have $L_1(\tilde{F}; \tilde{a}_n, \tilde{b}_n, z) \rightarrow L_1(\tilde{F}; \tilde{a}, \tilde{b}, z)$ as $n \rightarrow +\infty$. By the definition B-property, Section 5.1 and Lebesgue's dominating convergence theorem, we have

$$\lim_{n \rightarrow +\infty} \left| \int_U L_1(\tilde{F}; \tilde{a}_n, \tilde{b}_n, z) d\mu \right| = 0$$

when $\tilde{a} = \tilde{b}$, and

$$\lim_{n \rightarrow +\infty} \left| \int_U L_1(\tilde{F}; \tilde{a}_n, \tilde{b}_n, z) d\mu \right| = \left| \int_U L_1(\tilde{F}; \tilde{a}, \tilde{b}, z) d\mu \right| < +\infty$$

when $\tilde{a} \neq \tilde{b}$, which conflicts with the limit 6.2.1. \square

From the definition of the B-property, Proposition 5.9, Proposition 5.11, Proposition 6.2 and Lemma 6.6, we get the following proposition.

Proposition 6.7. *Suppose that F is the time-one map of an identity isotopy I on M , the measure $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont}, I}(F)$ and $\rho_{M, I}(\mu) = 0$. If one of the following three cases is satisfied*

- *I satisfies the B-property and $F \in \text{Diff}(M)$ (in particular $F \in \text{Diff}^1(M)$);*
- *I satisfies the B-property, the measure $\mu \in \mathcal{M}(F)$ has total support;*
- *I satisfies the B-property, the measure $\mu \in \mathcal{M}(F)$ is ergodic,*

then the action L_μ is uniformly bounded on $\text{Fix}_{\text{Cont}, I}(F)$.

Proof. By Proposition 6.2, we only need consider a compact set \tilde{P} of \tilde{M} such that \tilde{P} contains a fundamental domain of the covering transformation group G (see 1.4.3). \square

We now study the continuity of the actions l_μ and L_μ . In Example 8.3 of Appendix, we will construct an isotopy I and a measure $\mu \in \mathcal{M}(F)$ such that the time-one map F is a diffeomorphism (hence satisfies the WB-property) but not a C^1 -diffeomorphism and the measure μ has total support and no atoms on $\text{Fix}_{\text{Cont}, I}(F)$, while the action is not continuous. However, we have the following results.

Proposition 6.8. *Suppose that F is the time-one map of an isotopy I on M and the measure $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont}, I}(F)$. If one of the following three cases is satisfied*

- *I satisfies the B-property and $F \in \text{Diff}(M)$ (in particular $F \in \text{Diff}^1(M)$);*
- *I satisfies the B-property, the measure $\mu \in \mathcal{M}(F)$ has total support;*
- *I satisfies the B-property, the measure $\mu \in \mathcal{M}(F)$ is ergodic,*

then the action l_μ is continuous on $\text{Fix}(\tilde{F})$. As a consequence, if $\rho_{M, I}(\mu) = 0$, the action L_μ is continuous on $\text{Fix}_{\text{Cont}, I}(F)$.

Proof. We suppose that $\tilde{a} \in \text{Fix}(\tilde{F}) \setminus \{\infty\}$ and $\{\tilde{a}_n\}_{n \geq 1} \subset \text{Fix}(\tilde{F}) \setminus \{\tilde{a}, \infty\}$ satisfying $\tilde{a}_n \rightarrow \tilde{a}$ as $n \rightarrow +\infty$. We consider the value $i_\mu(\tilde{F}; \tilde{a}_n, \tilde{a})$. There exists a triangulation $\{\text{Cl}(U_i)\}_{i=1}^{+\infty}$ of $M \setminus \text{Fix}(F)$ such that, for every i , the interior U_i of $\text{Cl}(U_i)$ is an open free disk for F and satisfies $\mu(\partial U_i) = 0$.

By Lemma 6.5, we have $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$ in the case where $z \in \text{Fix}(F) \setminus \{\pi(\tilde{a})\}$, and $\lim_{n \rightarrow +\infty} L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$ in the case where $z \in \text{Rec}^+(F) \cap U_i$, for every i .

Choose a compact set $\tilde{P} \subset \tilde{M}$ such that $\tilde{a} \in \tilde{P}$ and $\{\tilde{a}_n\}_{n \geq 1} \subset \tilde{P}$. As before, when \tilde{a}' and \tilde{b}' are two distinct fixed points of \tilde{F} in \tilde{P} , we can always suppose that the path $\tilde{\gamma}$ that joins \tilde{a}' and \tilde{b}' is in \tilde{P} in this proof when we talk of the linking number $i(\tilde{F}; \tilde{a}', \tilde{b}', z)$. By the definition of B-property, Proposition 5.9, 5.11 and Lemma 6.6, we can suppose that there exists a number $N \geq 0$ such that

$$N = \sup_{n \geq 1} \left\{ |i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \mid z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}_n, \tilde{a}\}) \right\}.$$

By Lebesgue's dominating convergence theorem (the dominated function is N), we get

$$\lim_{n \rightarrow +\infty} \int_{\text{Fix}(F)} |i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu = 0.$$

It is then sufficient to prove that

$$\lim_{n \rightarrow +\infty} \int_{M \setminus \text{Fix}(F)} |i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu = 0.$$

Fix any $\epsilon > 0$. Since $\mu(\bigcup_{i=1}^{+\infty} U_i) = \mu(M \setminus \text{Fix}(F)) < +\infty$, there exists a positive integer N' such that

$$\mu\left(\bigcup_{N'+1}^{+\infty} U_i\right) < \frac{\epsilon}{2N'}.$$

For every pair $(\tilde{a}, \tilde{b}) \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ and each i , by Birkhoff Ergodic theorem, we have $\tau^*(\Phi(z)) = \tau^*(z)$ where $\tau^*(z)$ is the limit of the sequence $\{\tau_n(z)/n\}_{n \geq 1}$ and Φ is the first return map (see 1.3.2), and $L^*(\tilde{F}; \tilde{a}, \tilde{b}, \Phi(z)) = L^*(\tilde{F}; \tilde{a}, \tilde{b}, z)$. Hence, $i(\tilde{F}; \tilde{a}, \tilde{b}, \Phi(z)) = i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for μ -almost every point $z \in U_i$. Obviously, $|i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)|\tau(z) \in L^1(U_i, \mathbb{R}, \mu)$.

Therefore, for μ -almost every point $z \in U_i$, we have

$$\begin{aligned}
& \lim_{m \rightarrow +\infty} \frac{1}{m} \sum_{j=0}^{m-1} \left(\tau(\Phi^j(z)) \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, \Phi^j(z)) \right| \right) \\
&= \lim_{m \rightarrow +\infty} \left(\frac{1}{m} \sum_{j=0}^{m-1} \tau(\Phi^j(z)) \right) \cdot \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| \\
&= \tau^*(z) \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right|.
\end{aligned}$$

This implies that

$$(6.2.2) \quad \int_{U_i} \tau(z) \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu = \int_{U_i} \tau^*(z) \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu$$

for every i and every n .

Recall that, for every pair $(\tilde{a}, \tilde{b}) \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ and every i ,

$$L^*(\tilde{F}; \tilde{a}, \tilde{b}, z) = \lim_{m \rightarrow +\infty} \frac{1}{m} \sum_{j=1}^{m-1} L_1(\tilde{F}; \tilde{a}, \tilde{b}, \Phi^j(z))$$

exists for μ -almost every point $z \in U_i$. From Proposition 5.9 and Proposition 5.11, we have $\left| L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| \leq N\tau(z)$, which implies that $L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \in L^1(U_i, \mathbb{R}, \mu)$ for every i . Therefore, we have the following inequality (modulo subsets of measure zero of U_i)

$$\begin{aligned}
(6.2.3) \quad \left| L^*(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| &= \lim_{m \rightarrow +\infty} \frac{1}{m} \left| \sum_{j=0}^{m-1} (L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, \Phi^j(z))) \right| \\
&\leq \lim_{m \rightarrow +\infty} \frac{1}{m} \sum_{j=0}^{m-1} \left| L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, \Phi^j(z)) \right| \\
&\triangleq \left| L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right|^*.
\end{aligned}$$

The last definition and equation hold due to Birkhoff Ergodic theorem.

Moreover, we have

$$\begin{aligned}
\int_{\bigcup_{i=1}^{N'} U_i} \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu &\leq \sum_{i=1}^{N'} \int_{\bigcup_{k \geq 0} F^k(U_i)} \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \tau(z) \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \tau^*(z) \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \left| L^*(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu \\
&\leq \sum_{i=1}^{N'} \int_{U_i} \left| L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right|^* d\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \left| L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu \\
&\rightarrow 0 \quad (n \rightarrow +\infty).
\end{aligned}$$

The second equation holds since F preserve the measure μ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is the action of F . The third equation holds by Equation 6.2.2. The fourth equation is true because $i(\tilde{F}; \tilde{a}, \tilde{b}, z) = L^*(\tilde{F}; \tilde{a}, \tilde{b}, z)/\tau^*(z)$. The fifth inequality holds by the Inequality 6.2.3. The sixth equation holds due to Birkhoff Ergodic theorem. The last limit holds due to Lebesgue's dominating convergence theorem (the dominated function is $N\tau(z)$) and Lemma 6.5.

Therefore, there exists a positive number N'' such that when $n \geq N''$,

$$\int_{\bigcup_{i=1}^{N'} U_i} \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu < \frac{\epsilon}{2}.$$

Finally, when $n \geq N''$, we have

$$\begin{aligned}
\int_{M \setminus \text{Fix}(F)} \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu &= \int_{\bigcup_{i=1}^{N'} U_i} \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu + \int_{\bigcup_{N'+1}^{+\infty} U_i} \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2N} \cdot N \\
&= \epsilon.
\end{aligned}$$

We have completed the first statement.

Now we turn to prove the second statement. Let $a \in \text{Fix}_{\text{Cont}, I}(F)$ and $\{a_n\}_{n \geq 1} \subset \text{Fix}_{\text{Cont}, I}(F) \setminus \{a\}$ satisfying $a_n \rightarrow a$ as $n \rightarrow +\infty$. By Proposition 6.2, we only need to consider a lift \tilde{a} of a and a lift sequence $\{\tilde{a}_n\}_{n \geq 1} \subset \text{Fix}(\tilde{F})$ of $\{a_n\}_{n \geq 1}$ satisfying $\tilde{a}_n \rightarrow \tilde{a}$ as $n \rightarrow +\infty$. Then it follows from the first statement. \square

6.3. Action spectrum. In this section, we suppose that the action l_μ is well defined. Write \tilde{F} as the lift of F obtained by lifting I to an isotopy \tilde{I} to \tilde{M} starting $\text{Id}_{\tilde{M}}$.

Define the *action spectrum* of I as follows (up to an additive constant):

$$\sigma(\tilde{F}) = \{l_\mu(\tilde{F}; \tilde{z}) \mid z \in \text{Fix}(\tilde{F})\} \subset \mathbb{R} \cup \{\pm\infty\}.$$

By Corollary 5.14, the action spectrum of I is invariant under conjugation by an orientation preserving and measure preserving homeomorphism of M .

Define the *action width* of I as follows:

$$\text{width}(\tilde{F}) = \sup_{x, y \in \sigma(\tilde{F})} |x - y| = \sup_{\tilde{z}_1, \tilde{z}_2 \in \text{Fix}(\tilde{F})} i_\mu(\tilde{F}; \tilde{z}_1, \tilde{z}_2).$$

Moreover, if $\rho_{M,I}(\mu) = 0$, we can write the action spectrum of I as (up to an additive constant):

$$\sigma(\tilde{F}) = \{L_\mu(\tilde{F}; z) \mid z \in \text{Fix}_{\text{Cont}, I}(F)\} \subset \mathbb{R} \cup \{\pm\infty\}.$$

Furthermore, if L_μ is continuous (see Proposition 6.7 and Proposition 6.8), $\sigma(\tilde{F})$ is a compact set of \mathbb{R} and we can write the action width of I as:

$$\text{width}(\tilde{F}) = \max_{z_1, z_2 \in \text{Fix}_{\text{Cont}, I}(F)} I_\mu(\tilde{F}, z_1, z_2).$$

The following Theorem is the Arnold conjecture for surface homeomorphisms that is due to Matsumoto [Mat] (see also [Lec1]):

Theorem 6.9. *Let M be a compact surface with genus $g \geq 1$ and F be the time-one map of an identity isotopy I on M . We suppose that $\mu \in \mathcal{M}(F)$ has total support and $\rho_{M,I}(\mu) = 0$. Then there exist at least three contractible fixed points of F .*

On a closed surface, based on Theorem 6.9, we can get the following result that is a generalization of Lemma 2.8 that is proved in [Sz] by using Floer homology.

Proposition 0.2 *Let F be the time-one map of an identity isotopy I on a closed oriented surface M with $g \geq 1$. If I satisfies the WB-property and $F \in \text{Homeo}_*(M) \setminus \{\text{Id}_M\}$, $\mu \in \mathcal{M}(F)$ has total support, no atoms on $\text{Fix}_{\text{Cont}, I}(F)$ and $\rho_{M,I}(\mu) = 0$, then $\sharp\sigma(\tilde{F}) \geq 2$, that is, the action function L_μ is not constant.*

The proof of Proposition 0.2 will be divided two cases: the set $\text{Fix}_{\text{Cont}, I}(F)$ is finite and the set $\text{Fix}_{\text{Cont}, I}(F)$ is infinite. Firstly, let us prove the case where the set $\text{Fix}_{\text{Cont}, I}(F)$ is finite which is an easier case.

Proof of the case of Proposition 0.2 where the set $\text{Fix}_{\text{Cont}, I}(F)$ is finite.

We say that $X \subseteq \text{Fix}_{\text{Cont}, I}(F)$ is *unlinked* if there exists an isotopy $I' = (F'_t)_{t \in [0,1]}$ homotopic to I which fixes every point of X . Moreover, we say that X is a *maximal unlinked set*, if any set $X' \subseteq \text{Fix}_{\text{Cont}, I}(F)$ which strictly contains X is not unlinked.

In the proof of Theorem 6.9 ([Lec1, Theorem 10.1]), Le Calvez proved that there exists a maximal unlinked set $X \subseteq \text{Fix}_{\text{Cont}, I}(F)$ with $\sharp X \geq 3$ if $\sharp \text{Fix}_{\text{Cont}, I}(F) < +\infty$.

There exists an oriented topological foliation \mathcal{F} on $M \setminus X$ (or, equivalently, a singular oriented foliation \mathcal{F} on M with X equal to the singular set) such that, for all $z \in M \setminus X$, the trajectory $I(z)$ is homotopic to an arc γ joining z and $F(z)$ in $M \setminus X$ which is positively transverse to \mathcal{F} . That means that for every $t_0 \in [0, 1]$ there exists an open neighborhood $V \subset M \setminus X$ of $\gamma(t_0)$ and an orientation preserving homeomorphism $h : V \rightarrow (-1, 1)^2$ which sends the foliation \mathcal{F} on the horizontal foliation (oriented with x_1 increasing) such that the map $t \mapsto p_2(h(\gamma(t)))$ defined in a neighborhood of t_0 is strictly increasing where $p_2(x_1, x_2) = x_2$.

We can choose a point $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$ and a leaf λ containing z . Proposition 10.4 in [Lec1] states that the ω -limit set $\omega(\lambda) \in X$, the α -limit set $\alpha(\lambda) \in X$ and $\omega(\lambda) \neq \alpha(\lambda)$. Fix an isotopy I' homotopic to I that fixes $\omega(\lambda)$ and $\alpha(\lambda)$ and a lift $\tilde{\lambda}$ of λ that joins $\widetilde{\omega(\lambda)}$ and $\widetilde{\alpha(\lambda)}$. Let us now study the linking number $i(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z')$ for $z' \in \text{Rec}^+(F) \setminus X$ if it exists. Observe that for all $z' \in M \setminus X$, the trajectory $I'(z')$ is still homotopic to an arc that is positively transverse to \mathcal{F} . Hence, for all $z' \in \text{Rec}^+(F) \setminus X$ and disk U containing z' (here, we suppose that $U \cap \lambda = \emptyset$ by shrinking U and perturbing λ if necessary), we have

$$L_n(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z') = \tilde{\lambda} \wedge \tilde{\Gamma}_{I', z'}^n = \lambda \wedge \Gamma_{I', z'}^n \geq 0$$

for every $n \geq 1$, where \tilde{I}' is the lift of I' to \tilde{M} (refer to Section 4.1). Finally, we have

$$i(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z') \geq 0$$

for μ -almost every point $z' \in \text{Rec}^+(F) \setminus \{\omega(\lambda), \alpha(\lambda)\}$ (refer to Definition 4.1).

By the continuity of I' and the hypothesis on μ , there exists an open free disk U containing z such that $\mu(U) > 0$ and $L_1(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z') > 0$ when $z' \in U \cap \text{Rec}^+(F)$.

Similarly to the proof of Proposition 6.8, we have

$$\begin{aligned} I_\mu(\tilde{F}; \omega(\lambda), \alpha(\lambda)) &\geq \int_{\bigcup_{k \geq 0} F^k(U)} i(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\ &= \int_U \tau(z) i(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\ &= \int_U \tau^*(z) i(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\ &= \int_U L^*(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\ &= \int_U L_1(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\ &> 0. \end{aligned}$$

□

Before proving the case where the set $\text{Fix}_{\text{Cont}, I}(F)$ is infinite, let us state a recent result due to Jaulent [J]:

Theorem 6.10 (Jalent). *Let M be an oriented surface and F be the time-one map of an identity isotopy I on M . There exists a closed subset $X \subset \text{Fix}(F)$ and an isotopy I' joining $\text{Id}_{M \setminus X}$ to $F|_{M \setminus X}$ in $\text{Homeo}(M \setminus X)$ such that*

- (1) *For all $z \in X$, the loop $I(z)$ is homotopic to zero in M .*
- (2) *For all $z \in \text{Fix}(F) \setminus X$, the loop $I'(z)$ is not homotopic to zero in $M \setminus X$.*
- (3) *For all $z \in M \setminus X$, the trajectories $I(z)$ and $I'(z)$ are homotopic with fixed endpoints in M .*
- (4) *There exists an oriented topological foliation \mathcal{F} on $M \setminus X$ such that, for all $z \in M \setminus X$, the trajectory $I'(z)$ is homotopic to an arc γ joining z and $F(z)$ in $M \setminus X$ which is positively to \mathcal{F} .*

Moreover, the isotopy I' satisfies the following property:

- (5) *For all finite $Y \subset X$, there exists an isotopy I'_Y joining Id_M and F in $\text{Homeo}(M)$ which fixes Y such that, if $z \in M \setminus X$, the arc $I'(z)$ and $I'_Y(z)$ are homotopic in $M \setminus Y$. And if $z \in X \setminus Y$, the loop $I'_Y(z)$ is contractible in $M \setminus Y$.*

Proof of the case of Proposition 0.2 where the set $\text{Fix}_{\text{Cont}, I}(F)$ is infinite.

Suppose that X , I' and \mathcal{F} are respectively the closed contractible fixed points set, the isotopy and foliation as stated in Theorem 6.10. Obviously, $X \neq \emptyset$ and $\mu(M \setminus X) > 0$. Assume that X' is the union of the connected components of X that separates M . Write $M \setminus X' = \cup_{i=1}^n S_i$ where $n \geq 1$ and S_i are the F -invariant subsurfaces of M . By the definitions of S_i and I' , we have the following properties

- (A1): if S_i is a disk, then we have $(X \setminus \partial S_i) \cap S_i \neq \emptyset$ (by Proposition 3.1 and item (2) of Theorem 6.10);
- (A2): $\rho_{S_i, I'}(\mu) = 0 \in H_1(S_i, \mathbb{R})$ for every i (by the item (1) and (3) of Theorem 6.10).

It implies that the sum of the number of the connected component of ∂S_i and the number of the connected component of $X \cap S_i$ is greater than 2. Indeed, it is sufficient to prove the case when S_i is not a subsurface of sphere by A1. Identifying every connected component of ∂S_i as one point, we get a closed surface S'_i and an identity isotopy induced by I' , written still I' , which satisfy $\rho_{S'_i, I'}(\mu) = 0 \in H_1(S'_i, \mathbb{R})$ by A2. Using Theorem 6.9, we prove the claim.

Fix one subsurface S_i . Similarly to Proposition 0.2, we choose a point $z \in (\text{Rec}^+(F) \setminus \text{Fix}(F)) \cap S_i$ and a leaf $\lambda \in \mathcal{F}$ containing z . The proofs of Proposition 4.1 (page 150, when S_i is a disk or an annulus) and Proposition 6.1 (page 166, when S_i is not a subsurface of the sphere) in [Lec2] say that $\omega(\lambda)$ (resp., $\alpha(\lambda)$) is connected and is contained in a connected component of ∂S_i or a connected component of $X \cap S_i$, written $X_+(\lambda)$ (resp. $X_-(\lambda)$). Moreover, $X_-(\lambda) \neq X_+(\lambda)$. Choose a lift $\tilde{\lambda}$ of λ . We have to consider the following four cases: the set $\omega(\tilde{\lambda})$ or $\alpha(\tilde{\lambda})$ contains ∞ or not.

Take two points $a \in \alpha(\lambda)$ and $b \in \omega(\lambda)$. Let $Y = \{a, b\}$ and I'_Y be the isotopy as stated in Theorem 6.10. Suppose that \tilde{I}'_Y is the identity lift of I'_Y to \tilde{M} . Notice that

- (B1): if $z \in M \setminus X$, the arcs $I'(z)$ and $I'_Y(z)$ are homotopic in $M \setminus Y$ by item (5) of Theorem 6.10, and by item (4) of Theorem 6.10, $I'_Y(z)$ is homotopic to an arc γ joining z and $F(z)$ in $M \setminus Y$ which is positively transverse to \mathcal{F} ;

(B2): if $z \in X \setminus Y$, then $\gamma \wedge I'_Y(z) = 0$ by the item (5) of Theorem 6.10 where γ is any path from a to b .

If both $\alpha(\tilde{\lambda})$ and $\omega(\tilde{\lambda})$ do not contain ∞ , replacing a by $\alpha(\lambda)$, b by $\omega(\lambda)$ and I' by I'_Y in the proof of Proposition 0.2, then we can get $I_\mu(\tilde{F}; a, b) > 0$.

We suppose now that at least one of $\alpha(\tilde{\lambda})$ and $\omega(\tilde{\lambda})$ contains ∞ . Recall that \tilde{d} is the distance of \tilde{M} induced by a distance d of M which is induced by a Riemannian metric on M . Define $\text{dist}(\tilde{z}, \tilde{C}) = \inf_{\tilde{c} \in \tilde{C}} \tilde{d}(\tilde{z}, \tilde{c})$ if $\tilde{z} \in \tilde{M}$ and $\tilde{C} \subset \tilde{M}$. Take a sequence $\{(\tilde{a}_m, \tilde{b}_m)\}_{m \geq 1}$ such that

- $\pi(\tilde{a}_m) = a$ and $\pi(\tilde{b}_m) = b$, if $\alpha(\tilde{\lambda})$ (resp. $\omega(\tilde{\lambda})$) does not contain ∞ , we set $\tilde{a}_m = \tilde{a}$ (resp. $\tilde{b}_m = \tilde{b}$) for every m where $\tilde{a} \in \pi^{-1}(a) \cap \alpha(\tilde{\lambda})$ (resp. $\tilde{b} \in \pi^{-1}(b) \cap \omega(\tilde{\lambda})$);
- $\lim_{m \rightarrow +\infty} \text{dist}(\tilde{a}_m, \tilde{\lambda}) = 0$ and $\lim_{m \rightarrow +\infty} \text{dist}(\tilde{b}_m, \tilde{\lambda}) = 0$.

For every m , suppose that \tilde{c}_m (resp. \tilde{c}'_m) is a point of $\tilde{\lambda}$ such that $\tilde{d}(\tilde{a}_m, \tilde{c}_m) = \text{dist}(\tilde{a}_m, \tilde{\lambda})$ (resp. $\tilde{d}(\tilde{b}_m, \tilde{c}'_m) = \text{dist}(\tilde{b}_m, \tilde{\lambda})$). Obviously, if $\alpha(\tilde{\lambda})$ (resp. $\omega(\tilde{\lambda})$) does not contain ∞ , then $\tilde{c}_m = \tilde{a}_m = \tilde{a}$ (resp. $\tilde{c}'_m = \tilde{b}_m = \tilde{b}$) and $\text{dist}(\tilde{a}_m, \tilde{\lambda}) = 0$ (resp. $\text{dist}(\tilde{b}_m, \tilde{\lambda}) = 0$). Choose a simple path \tilde{l}_m (resp. \tilde{l}'_m) from \tilde{a}_m (resp. \tilde{c}'_m) to \tilde{c}_m (resp. \tilde{b}_m) such that the length of \tilde{l}_m (resp. \tilde{l}'_m) is $\text{dist}(\tilde{a}_m, \tilde{\lambda})$ (resp. $\text{dist}(\tilde{b}_m, \tilde{\lambda})$). Here, we set the simple path is empty set if its length is 0. Let $\tilde{\gamma}_m = \tilde{l}_m \tilde{\lambda}_m \tilde{l}'_m$ where $\tilde{\lambda}_m$ is the sub-path of $\tilde{\lambda}$ from \tilde{c}_m to \tilde{c}'_m . Then $\tilde{\gamma}_m$ is a path from \tilde{a}_m to \tilde{b}_m .

We know that, for every $m \geq 1$, the linking number $i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z')$ exists for μ -almost every $z' \in M \setminus \{a, b\}$. Hence, the linking number $i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z')$ exists on a full measure subset of $M \setminus \{a, b\}$ for all m . By the property B2 above, we have $i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') = 0$ if $z' \in X \setminus \{a, b\}$. We now claim that $\liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') \geq 0$ for μ -almost every $z' \in \text{Rec}^+(F) \setminus X$.

Fix one point $z' \in \text{Rec}^+(F) \setminus X$ and choose a disk U containing z' (here again, we suppose that $U \cap \lambda = \emptyset$). By the property B1 and the construction of $\tilde{\gamma}_m$, for every $n \geq 1$, there exists $m(z', n) \in \mathbb{N}$ such that when $m \geq m(z', n)$, the value

$$L_n(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') = \tilde{\gamma}_m \wedge \tilde{\Gamma}_{I'_Y, z'}^n = \pi(\tilde{\gamma}_m) \wedge \Gamma_{I'_Y, z'}^n$$

is constant with regard to m and

$$(6.3.1) \quad L_n(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') \geq 0.$$

We now suppose that

$$\mu\{z' \in \text{Rec}^+(F) \setminus X \mid \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') < 0\} > 0.$$

There exists a small number $c > 0$ such that

$$(6.3.2) \quad \mu\{z' \in \text{Rec}^+(F) \setminus X \mid \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') < -c\} > c.$$

Write $E = \{z' \in \text{Rec}^+(F) \setminus X \mid \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') < -c\}$. Fix a point $z' \in E$ and a disk U containing z' as before. By taking subsequence if necessary, we may suppose that

$$-\infty \leq \lim_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') < -c.$$

Then there exists $N(z') \in \mathbb{N}$ such that when $m \geq N(z')$, we have

$$i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') = \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z')}{\tau_n(z')} < -c.$$

Fix $m_0 \geq N(z')$. There exists $n(z', m_0) \in \mathbb{N}$ such that when $n \geq n(z', m_0)$, we have

$$\frac{L_n(\tilde{F}; \tilde{a}_{m_0}, \tilde{b}_{m_0}, z')}{\tau_n(z')} < -c.$$

Fix $n_0 \geq n(z', m_0)$. Then, we have

$$L_{n_0}(\tilde{F}; \tilde{a}_{m_0}, \tilde{b}_{m_0}, z') < -c\tau_{n_0}(z').$$

By the inequality 6.3.1, there exists $m(z', n_0) > m_0$ such that, when $m \geq m(z', n_0)$, we have

$$L_{n_0}(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') \geq 0.$$

Fix $m_1 \geq m(z', n_0)$. There exists $n(z', m_1) > n_1$ such that when $n \geq n(z', m_1)$, we have

$$\frac{L_n(\tilde{F}; \tilde{a}_{m_1}, \tilde{b}_{m_1}, z')}{\tau_n(z')} < -c.$$

Fix $n_1 \geq n(z', m_1)$. Then, we have

$$L_{n_1}(\tilde{F}; \tilde{a}_{m_1}, \tilde{b}_{m_1}, z') < -c\tau_{n_1}(z').$$

By induction, we can construct a sequence $\{(m_i, n_i)\}_{i \geq 0} \subset \mathbb{N} \times \mathbb{N}$ such that

(C1): $\{m_i\}_{i \geq 0}$ and $\{n_i\}_{i \geq 0}$ are strictly increasing sequences;

(C2): for every $i \geq 0$,

$$L_{n_i}(\tilde{F}; \tilde{a}_{m_i}, \tilde{b}_{m_i}, z') < -c\tau_{n_i}(z') \quad \text{and} \quad L_{n_i}(\tilde{F}; \tilde{a}_{m_{i+1}}, \tilde{b}_{m_{i+1}}, z') \geq 0.$$

As the positively transverse property of \mathcal{F} , it is easy to see that the negative part of $L_{n_i}(\tilde{F}; \tilde{a}_{m_i}, \tilde{b}_{m_i}, z')$ only comes from the intersection l_{m_i} or l'_{m_i} , or both of them with the curve $\tilde{\Gamma}_{I'_Y, z'}^n$ in the case where $\alpha(\tilde{\lambda})$ or $\omega(\tilde{\lambda})$ contains ∞ , or both of them contain ∞ .

We deal with the case where both $\alpha(\tilde{\lambda})$ and $\omega(\tilde{\lambda})$ contain ∞ , and other cases follow similarly. By the item (5) of Theorem 6.10, it is easy to see that there is a positive integer N such that the number of times that $I'_Y(x)$ rotates around a (resp. b) is less than N when x is close to a (resp. b). Since I'_Y fixes a and b , the measure μ has no atoms on $\text{Fix}_{\text{Cont}, I}(F)$, the construction of $\tilde{\lambda}_m$, and the property C2, there must be an open disks sequence $\{U_i^a\}_{i \geq 0}$ that contains the set $(I'_Y)^{-1}(\pi(\tilde{l}_{m_i})) = \cup_{y \in \pi(\tilde{l}_{m_i})} (I'_Y)^{-1}(y)$ and an open disks sequence $\{U_i^b\}_{i \geq 0}$ that contains the set $(I'_Y)^{-1}(\pi(\tilde{l}'_{m_i})) = \cup_{y \in \pi(\tilde{l}'_{m_i})} (I'_Y)^{-1}(y)$ satisfying

(D1): $U_{i+1}^a \subset U_i^a$ (resp. $U_{i+1}^b \subset U_i^b$) and $\mu(U_i^a) \rightarrow 0$ (resp. $\mu(U_i^b) \rightarrow 0$) as $i \rightarrow +\infty$;

(D2): for every $i \geq 0$,

$$\frac{1}{\tau_{n_i}(z')} \sum_{j=0}^{\tau_{n_i}(z')-1} \chi_{U_i^a} \circ F^j(z') > \frac{c}{2N} \quad \text{or} \quad \frac{1}{\tau_{n_i}(z')} \sum_{j=0}^{\tau_{n_i}(z')-1} \chi_{U_i^b} \circ F^j(z') > \frac{c}{2N},$$

where χ_U is the characteristic function of $U \subset M$.

Denote by $\chi_U^*(x)$ the limit function of $\frac{1}{n} \sum_{j=0}^{n-1} \chi_U \circ F^j(x)$ as $n \rightarrow +\infty$ for μ -almost every $x \in M$ (by Birkhoff Ergodic theorem). By the property D2 and the inequality 6.3.2, we have

$$\mu(\{x \in \text{Rec}^+(F) \setminus X \mid \chi_{U_i^a}^*(x) \geq \frac{c}{2N} \quad \text{or} \quad \chi_{U_i^b}^*(x) \geq \frac{c}{2N}\}) > c$$

for every i . This implies that $\int_M (\chi_{U_i^a}^*(x) + \chi_{U_i^b}^*(x)) d\mu \geq \frac{c^2}{2N} > 0$ for every i . On the other hand, by Birkhoff Ergodic theorem and the property D1, we have

$$\int_M (\chi_{U_i^a}^*(x) + \chi_{U_i^b}^*(x)) d\mu = \int_M (\chi_{U_i^a}(x) + \chi_{U_i^b}(x)) d\mu = \mu(U_i^a) + \mu(U_i^b) \rightarrow 0$$

as $i \rightarrow +\infty$, which is impossible.

Finally, we get

$$(6.3.3) \quad \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') \geq 0$$

for μ -almost every point $z' \in \text{Rec}^+(F) \setminus \{a, b\}$.

By the continuity of I_Y' and the hypothesis on μ , there exists an open free disk U containing z such that $\mu(U) > 0$ and

$$(6.3.4) \quad \lim_{m \rightarrow +\infty} L_1(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') > 0$$

when $z' \in U \cap \text{Rec}^+(F)$.

As the rotation vector of μ vanishes, by Proposition 6.2, the inequalities 6.3.3, 6.3.4 and Fatou Lemma, we have

$$\begin{aligned}
I_\mu(\tilde{F}; a, b) &= \lim_{m \rightarrow +\infty} i_\mu(\tilde{F}; \tilde{a}_m, \tilde{b}_m) \\
&= \lim_{m \rightarrow +\infty} \int_{M \setminus \{a, b\}} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\
&\geq \int_{M \setminus \{a, b\}} \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\
&\geq \int_{\bigcup_{k \geq 0} F^k(U)} \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\
&= \int_U \liminf_{m \rightarrow +\infty} \tau(z) i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\
&= \int_U \liminf_{m \rightarrow +\infty} \tau^*(z) i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\
&= \int_U \liminf_{m \rightarrow +\infty} L^*(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\
&= \int_U \liminf_{m \rightarrow +\infty} L_1(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\
&> 0.
\end{aligned}$$

□

As a immediate consequence of Proposition 0.2, we have the following corollary.

Corollary 6.11. *Suppose that F is the time-one map of an identity isotopy I on a closed oriented surface M with $g \geq 1$, $\mu \in \mathcal{M}(F)$ has total support, no atoms on $\text{Fix}_{\text{Cont}, I}(F)$ and $\rho_{M, I}(\mu) = 0$. If $F \in \text{Diff}_*(M) \setminus \{\text{Id}_M\}$, then $\sharp\sigma(\tilde{F}) \geq 2$.*

From Proposition 6.2 and Proposition 0.2, we can get the following generalization of Theorem 2.1. C in [P1] on closed surface.

Corollary 0.3 *Let F be the time-one map of an identity isotopy I on a closed oriented surface M with $g > 1$. If I satisfies the WB-property and $F \in \text{Homeo}_*(M) \setminus \{\text{Id}_M\}$, $\mu \in \mathcal{M}(F)$ has total support and no atoms on $\text{Fix}_{\text{Cont}, I}(F)$, then $\sharp\sigma(\tilde{F}) \geq 2$.*

Proof. If $\rho_{M, I}(\mu) = 0$, by Proposition 0.2 and Proposition 6.11, there exist two distinct contractible fixed points a and b of F such that $I_\mu(\tilde{F}; a, b) \neq 0$, thus for any their lifts \tilde{a} and \tilde{b} we have $i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = I_\mu(\tilde{F}; a, b) \neq 0$.

If $\rho_{M, I}(\mu) \neq 0$, by the proof of Proposition 6.2, there exists $\alpha \in G^*$ such that $\varphi(\alpha) \wedge \rho_{M, I}(\mu) \neq 0$ where φ is the Hurewitz homomorphism from G to $H_1(M, \mathbb{Z})$. By Lefschetz-Nielsen's formula, we know that $\text{Fix}_{\text{Cont}, I}(F) \neq \emptyset$. Choose $a \in \text{Fix}_{\text{Cont}, I}(F)$, and an isotopy I' homotopic to I that fixes a . For any lifts \tilde{a} and $\alpha(\tilde{a})$ of a , we get that $i_\mu(\tilde{F}; \tilde{a}, \alpha(\tilde{a})) = \varphi(\alpha) \wedge \rho_{M, I}(\mu) \neq 0$. We have completed the proof. □

Let us now give two examples to see what will happen when $\text{Supp}(\mu) \neq M$.

Example 6.12. Consider the following smooth identity isotopy on \mathbb{R}^2 : $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]} : (x, y) \mapsto (x + \frac{t}{2\pi} \cos(2\pi y), y + \frac{t}{2\pi} \sin(2\pi y))$. It induces an identity smooth isotopy $I = (F_t)_{t \in [0,1]}$ on \mathbb{T}^2 . Let μ have constant density on $\{(x, y) \in \mathbb{T}^2 \mid y = 0 \text{ or } y = \frac{1}{2}\}$ and vanish on elsewhere. Obviously, $\rho_{\mathbb{T}^2, I}(\mu) = 0$ but $\text{Fix}_{\text{Cont}, I}(F_1) = \emptyset$.

The example 6.12 tells us that there is no sense to talk about the action function when $g = 1$ and $\text{Supp}(\mu) \neq M$. The following example belongs to Le Calvez [Lec1, page 73] who mentioned me that this example implies that Proposition 0.2 is not true anymore in the case where $g > 1$ and $\text{Supp}(\mu) \neq M$. For convenience of readers, we provide the example.

Example 6.13. Let M be the closed orientated surface with $g = 2$ and L be the morse function on M which has six critical points z_1, \dots, z_6 such that the points z_i ($2 \leq i \leq 5$) are saddle and the six critical values $L(z_i) = a_i$ ($1 \leq i \leq 6$) satisfy $a_1 < a_2 < \dots < a_6$ (see Figure 1). Fix $b_2 \in]a_2, a_3[$ and $b_4 \in]a_4, a_5[$. Denote by C_1, C_2 the two connected components of $L^{-1}(\{b_2\})$ and by C_3, C_4 the two connected components of $L^{-1}(\{b_4\})$. Fix $b'_2 \in]a_2, b_2[$ and $b'_4 \in]b_4, a_5[$ and modify the Hamiltonian vector field with regard to L on the closed annulus $L^{-1}([a_1, b'_2])$ and $L^{-1}([b'_4, a_6])$ to construct two components of Reeb and obtain a vector field ξ on M such that the critical points z_3 and z_4 of L are the only two singular points of ξ .

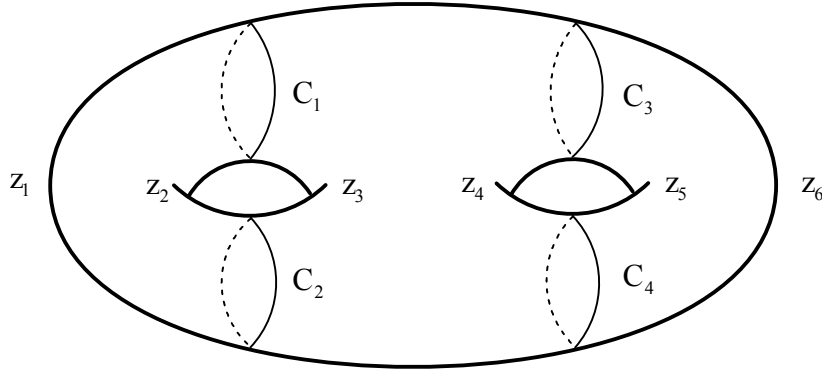


FIGURE 1. The counterexample when $\text{Supp}(\mu) \neq M$ and $g = 2$

The vector field ξ induces a natural orientation on the circles C_i . For two distinct points z and z' of C_i , denote by $[z, z']_i$ the segment which joins z and z' on C_i with the orientation. Choose the points as follows (see Figure 2)

- $z_{1,3}, z_{3,1}, z_{1,4}, z_{4,1}$ on C_1 whose cyclic order is induced by the orientation of C_1 ;
- $z_{2,4}, z_{4,2}$ on C_2 ;
- $z'_{1,3}, z'_{3,1}$ on C_3 ;
- $z'_{1,4}, z'_{4,1}, z'_{4,2}, z'_{2,4}$ on C_4 whose cyclic order is induced by the orientation of C_4 .

We can construct the following disjoint segments in $M \setminus \{z_3, z_4\}$

- an oriented segment $\gamma_{1,3}$ from $z_{1,3}$ to $z'_{1,3}$;
- an oriented segment $\gamma_{3,1}$ from $z'_{3,1}$ to $z_{3,1}$;

- an oriented segment $\gamma_{1,4}$ from $z_{1,4}$ to $z'_{1,4}$;
- an oriented segment $\gamma_{4,1}$ from $z'_{4,1}$ to $z_{4,1}$;
- an oriented segment $\gamma_{2,4}$ from $z_{2,4}$ to $z'_{2,4}$;
- an oriented segment $\gamma_{4,2}$ from $z'_{4,2}$ to $z_{4,2}$.

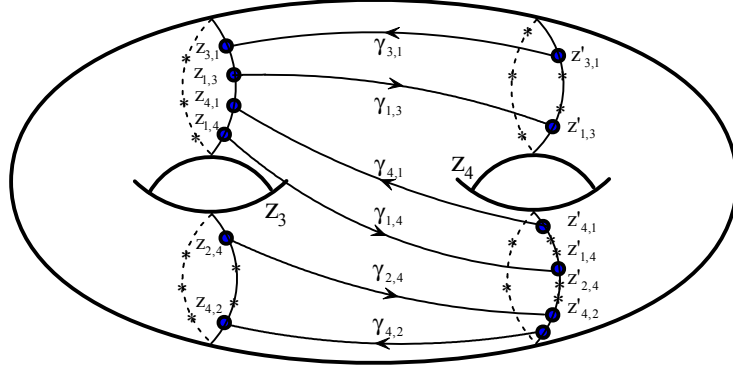


FIGURE 2. The counterexample when $\text{Supp}(\mu) \neq M$ and $g = 2$

The following three closed curves

$$\gamma_{1,3} \cup \gamma_{3,1} \cup [z_{1,3}, z_{3,1}]_1 \cup [z'_{1,3}, z'_{3,1}]_3,$$

$$\gamma_{1,4} \cup \gamma_{4,1} \cup [z_{1,4}, z_{4,1}]_1 \cup [z'_{1,4}, z'_{4,1}]_4,$$

$$\gamma_{2,4} \cup \gamma_{4,2} \cup [z_{4,2}, z_{2,4}]_2 \cup [z'_{4,2}, z'_{2,4}]_4$$

bound three disjoint closed disks in $L^{-1}([b_2, b_4])$. Up to multiplying the vector field by a strict positive function, denote by $(\theta_t)_{t \in \mathbb{R}}$ the flow with regard to ξ , we can suppose that

- the segments $\theta_t(\gamma_{i,j})$, $t \in [-1, 1]$ are pairwise disjoint for all (i, j) ;
- the segments $\theta_t(\gamma_{i,j})$ and $\theta_{t'}(\gamma_{i',j'})$ are pairwise disjoint for all $(t, t') \in [-1, 1]^2$ if $(i, j) \neq (i', j')$;
- $z'_{3,1} = \theta_3(z'_{1,3})$, $z'_{1,3} = \theta_4(z'_{3,1})$;
- $z_{1,4} = \theta_4(z_{3,1})$, $z_{1,3} = \theta_4(z_{4,1})$;
- $z'_{4,1} = \theta_3(z'_{1,4})$, $z'_{4,2} = \theta_3(z'_{4,1})$, $z'_{1,4} = \theta_3(z'_{2,4})$;
- $z_{2,4} = \theta_3(z_{4,2})$, $z_{4,2} = \theta_4(z_{2,4})$.

We now fix neighborhoods $U_{i,j}$ of $\gamma_{i,j}$ such that the six neighborhoods are pairwise disjoint and

$$U_{i,j} \cap L^{-1}([b_2, b_4]) \subset \bigcup_{-1 \leq t \leq 1} \theta_t(\gamma_{i,j}).$$

We consider an isotopy $(G_t)_{t \in [0,1]}$ whose support is in the union of the six neighborhoods $U_{i,j}$ such that

- the arc $t \mapsto G_t(z_{i,j})$ is the segment $\gamma_{i,j}$, if $i \in \{1, 2\}$;
- the arc $t \mapsto G_t(z'_{i,j})$ is the segment $\gamma_{i,j}$, if $i \in \{3, 4\}$.

After that, we define an isotopy $I = (F_t)_{t \in [0,1]}$ as follows

$$F_t = \begin{cases} \theta_{4t} & \text{if } t \in [0, 1/2]; \\ G_{2t-1} \circ \theta_2 & \text{if } t \in [1/2, 1]. \end{cases}$$

By construction above, we get the point $z'_{1,3}$ is a periodic point of F_1 with periodic 20 and the arc $\prod_{0 \leq i \leq 19} I(F_1^i(z'_{1,3}))$ is homologic to the sum of circles C_i in $M \setminus \{z_3, z_4\}$

and hence homologic to zero. Let the measure $\mu = \frac{1}{20} \sum_{i=0}^{19} \delta_{F_1^i(z'_{1,3})}$, where δ_z is the Dirac measure. The points z_3 and z_4 are the only two contractible fixed points of F_1 . Obviously, $\rho_{M,I}(\mu) = 0$ and hence the action function is constant.

7. APPLICATION TO THE GROUP OF CONSERVATIVE DIFFEOMORPHISMS

Fix a Borel finite measure μ on M , and denote by $\text{Homeo}_*(M, \mu)$ the subgroup of $\text{Homeo}_*(M)$ whose element preserves the measure μ . Denote by $\text{Hameo}(M, \mu)$ the subset of $\text{Homeo}_*(M, \mu)$ whose elements satisfy furthermore that $\rho_{M,I}(\mu) = 0$. When M is a compact surface, Franks (see [Fr2] for the details) has proved that $\text{Hameo}(M, \mu)$ forms a group. In this section, we suppose that the support of μ is M .

Denote by $\text{Ham}^1(M, \mu)$ the group $\text{Hameo}(M, \mu) \cap \text{Diff}^1(M)$ and by $\text{Diff}_*^1(M, \mu)$ the group $\text{Homeo}_*(M, \mu) \cap \text{Diff}^1(M)$. For convenience, we write M_g the oriented closed surface with the genus $g \geq 1$.

In the first part of this section, we will discuss the torsion in the group $\text{Hameo}(\mathbb{T}^2, \mu)$ and $\text{Homeo}_*(M_g, \mu)$ with $g > 1$. Moreover, by using a result of Fathi [Fa], we can get an indirect proof about periodic homeomorphisms of surfaces, that is, the group $\text{Homeo}_*(M_g)$ with $g > 1$ is torsion free. In the second part, we will study the distortion in the group $\text{Ham}^1(\mathbb{T}^2, \mu)$ and the group $\text{Diff}_*^1(M_g, \mu)$ with $g > 1$. The second part links to Zimmer conjecture on closed oriented surfaces.

7.1. The absence of torsion in $\text{Hameo}(\mathbb{T}^2, \mu)$ and $\text{Homeo}_*(M_g, \mu)$ with $g > 1$. From Corollary 6.3 and Proposition 0.2, we have the following proposition, which is a generalization of Proposition 2.6. A in [P1].

Proposition 0.4 *Under the same hypotheses as Proposition 0.2, there exists a constant $C > 0$ such that $\text{width}(\tilde{F}^n) \geq C \cdot n$ for every $n \geq 1$.*

The proposition 0.4 implies that the groups $\text{Hameo}(M_g, \mu)$ ($g \geq 1$) is torsion free.

From Corollary 5.12 and Corollary 0.3, we then get the following conclusion:

Proposition 0.5 *Under the same hypotheses as Corollary 0.3, there exists a constant $C > 0$ such that $\text{width}(\tilde{F}^n) \geq C \cdot n$ for every $n \geq 1$.*

Denote by $\mathcal{IS}_*(M)$ the group of all identity isotopies $I = (F_t)_{t \in [0,1]}$ on M , where the composition is given by Equation 1.1.1. Denote by $\mathcal{IS}_*(M, \mu)$ the subgroup of $\mathcal{IS}_*(M)$ whose element $(F_t)_{t \in [0,1]} \in \mathcal{IS}_*(M, \mu)$ satisfies $(F_t)_*\mu = \mu$ for all t . We say that two identity isotopies $(F_t)_{t \in [0,1]}$ and $(G_t)_{t \in [0,1]}$ are homotopic with fixed extremities if $F_1 = G_1$ and there exists a continuous map $[0, 1]^2 \rightarrow \text{Homeo}(M)$, $(t, s) \mapsto H_{t,s}$ such that $H_{0,s} =$

Id_M , $H_{1,s} = F_1 = G_1$, $H_{t,0} = F_t$ and $H_{t,1} = G_t$. The homotopic relation is an equivalence relation on $\mathcal{IS}_*(M)$ (resp. $\mathcal{IS}_*(M, \mu)$). We note the set of equivalence classes by $\mathcal{H}_*(M)$ (resp. $\mathcal{H}_*(M, \mu)$). It is not difficult to see that $\mathcal{H}_*(M)$ and $\mathcal{H}_*(M, \mu)$ are groups. Indeed, $\mathcal{H}_*(M)$ is the universal covering space of $\text{Homeo}_*(M)$ (see [Fa]).

We can divide the group $\mathcal{H}_*(M, \mu)$ into two sets by whether an element $I \in \mathcal{H}_*(M, \mu)$ satisfying the WB-property. In the subset of $\mathcal{H}_*(M, \mu)$ whose elements satisfy the WB-property, denoted by \mathcal{W} , we can continue to divide this set into two sets by whether an element $I \in \mathcal{W}$ satisfying $\rho_{M,I}(\mu) = 0$.

Given $I \in \mathcal{H}_*(M, \mu)$. If $I \notin \mathcal{W}$, by Definition 1.7, there exist two fixed points \tilde{a} and \tilde{b} of \tilde{F} such that $i(\tilde{F}; \tilde{a}, \tilde{b}) \neq 0$. By Equation 1.4.2, we have $i(\tilde{F}^n; \tilde{a}, \tilde{b}) = n \cdot i(\tilde{F}; \tilde{a}, \tilde{b}) \neq 0$ for every $n \in \mathbb{N}$, that is, $|i(\tilde{F}^n; \tilde{a}, \tilde{b})| \geq n$. If $\rho_{M,I}(\mu) \neq 0$, by the morphism property of $\rho_{M,\cdot}(\mu) : \mathcal{H}_*(M, \mu) \rightarrow H_1(M, \mathbb{R})$: $\rho_{M,II'}(\mu) = \rho_{M,I}(\mu) + \rho_{M,I'}(\mu)$, we have $\|\rho_{M,I^n}(\mu)\|_{H_1(M, \mathbb{R})} \geq n$. From Proposition 0.4 and Proposition 0.5, we get

Corollary 7.1. *The groups $\mathcal{H}_*(M_g, \mu)$ ($g \geq 1$) are torsion free.*

In [Fa], Fathi proved the following result: if M^n is a compact connected manifold without boundary of dimension $n \geq 1$, and μ is a finite measure on M^n without atoms and with total support, then the inclusion

$$\text{Homeo}(M^n, \mu) \hookrightarrow \text{Homeo}(M^n)$$

is a weak homotopy equivalence, that is, it induces isomorphisms on all homotopy groups.

We suppose now that the measure in Corollary 7.1 has no atoms on M_g . Observing that $\pi_1(\text{Homeo}_*(\mathbb{T}^2)) \simeq \mathbb{Z}^2$ (see [Ham1]) and $\pi_1(\text{Homeo}_*(M_g)) \simeq 0$ ($g > 1$) (see [Ham2]), we have

Corollary 0.6 *The groups $\text{Homeo}(\mathbb{T}^2, \mu)$, $\text{Homeo}_*(M_g)$ ($g > 1$) are torsion free.*

7.2. The absence of distortion in $\text{Ham}^1(\mathbb{T}^2, \mu)$ and $\text{Diff}_*^1(M_g, \mu)$ with $g > 1$. In 2002, Polterovich [P1] showed us a Hamiltonian version of the Zimmer program (see [Z, P2] for the detail) dealing with actions of lattices. It is achieved by using the classical action defined in symplectic geometry, the symplectic filling function (see Section 1.2 in [P1]), and a result of Schwarz [Sz] about the action function being non-constant which has been proved by using Floer homology. In 2003, Franks and Handel [FH1] developed the Thurston theory of normal forms for surface homeomorphisms with finite fixed sets. In 2006, they [FH2] used the generalized normal form to give a more general version (the map is a C^1 -diffeomorphism and the measure is a Borel finite measure) of the Zimmer program on the closed oriented surfaces. We recommend the reader a survey by Fisher [Fi] for the recent progress of Zimmer program. We will give an alternative proof of the C^1 -version of the Zimmer conjecture on surfaces when the measure is a Borel finite measure with full support.

Suppose that F is a C^1 -diffeomorphism of M_g ($g \geq 1$) which is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on M_g and \tilde{F} is the time-one map of the lifted identity isotopy $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]}$ on the universal cover \tilde{M} of M_g .

Lemma 7.2. *If there exist two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , and a point $z_* \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$ such that $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ exists and is not zero, then*

$$\|F^n\|_{\mathcal{G}} \succeq \sqrt{n}.$$

Proof. If $z_* \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ exist, by Lemma 8.1 and Proposition 4.3, we know that $z_* \in \text{Rec}^+(F^n) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, that $i(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*)$ exists and that $i(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) = ni(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ for all $n \geq 1$.

Let $N(n) = \|F^n\|_{\mathcal{G}}$. Assume that there exist identity isotopies $I_i = (F_{i,t})_{t \in [0,1]} \subset \text{Diff}^1(M)$ ($1 \leq i \leq s$) such that, for every $n \geq 1$, we have the isotopy I^n is homotopic to the isotopy $\prod_{j=1}^{N(n)} I_{i_j}^{\epsilon_j} = (F_t^{(n)})_{0 \leq t \leq 1} \triangleq I^{(n)}$, where $i_j \in \{1, 2, \dots, s\}$, $\epsilon_j \in \{-1, 1\}$ ($j = 1, 2, \dots, N(n)$) and

$$F_t^{(n)}(z) = F_{i_k, N(n)t - (k-1)}^{\epsilon_{i_k}}(F_{i_{k-1}, 1}^{\epsilon_{i_{k-1}}} \circ \dots \circ F_{i_1, 1}^{\epsilon_{i_1}}(z)), \quad \text{if } \frac{k-1}{N(n)} \leq t \leq \frac{k}{N(n)}.$$

Let $\tilde{I}_i = (\tilde{F}_{i,t})_{t \in [0,1]}$ ($1 \leq i \leq s$) and $\tilde{I}^{(n)} = (\tilde{F}_t^{(n)})_{0 \leq t \leq 1}$ be the lifts of I_i ($1 \leq i \leq s$) and $(F_t^{(n)})_{0 \leq t \leq 1}$ to \tilde{M} respectively. Identify the sphere $\tilde{M} \cup \{\infty\}$ as the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Again, for simplicity, we can suppose that $\tilde{a} = 0$ and $\tilde{b} = 1$.

Fix $n \geq 1$. Using the method of Lemma 1.2, we can get the isotopy $\tilde{I}'^{(n)} = (\tilde{F}_t'^{(n)})_{0 \leq t \leq 1}$ which fixes 0, 1 and is an isotopy on \tilde{M} from $\text{Id}_{\tilde{M}}$ to \tilde{F}^n , where

$$(7.2.1) \quad \tilde{F}_t'^{(n)}(\tilde{z}) = \frac{\tilde{F}_t^{(n)}(\tilde{z}) - \tilde{F}_t^{(n)}(0)}{\tilde{F}_t^{(n)}(1) - \tilde{F}_t^{(n)}(0)}.$$

Let $\tilde{\gamma} = \{0 \leq r \leq 1\}$ be the straight line from 0 to 1. If $\tilde{I}'^{(n)}(\tilde{z}) \cap \tilde{\gamma} \neq \emptyset$ for some point $\tilde{z} \in \tilde{M} \setminus \{0, 1\}$, then there exist $t_0 \in [0, 1]$ and $r_0 \in]0, 1[$ such that $\tilde{F}_{t_0}'^{(n)}(\tilde{z}) = r_0$, that is

$$(7.2.2) \quad \tilde{F}_{t_0}^{(n)}(\tilde{z}) - \tilde{F}_{t_0}^{(n)}(0) = r_0(\tilde{F}_{t_0}^{(n)}(1) - \tilde{F}_{t_0}^{(n)}(0)).$$

Let

$$C = \max_{i \in \{1, \dots, s\}} \sup_{t \in [0,1], \tilde{z} \in \tilde{M}} \tilde{d}(\tilde{F}_{i,t}, \tilde{z}).$$

We have

$$(7.2.3) \quad \left| \tilde{F}_t^{(n)}(1) - \tilde{F}_t^{(n)}(0) \right| \leq 2CN(n) + 1$$

and

$$\left| \tilde{F}_t^{(n)}(\tilde{z}) - \tilde{F}_t^{(n)}(0) \right| \geq |\tilde{z}| - 2CN(n)$$

for all $t \in [0, 1]$. Hence when $|\tilde{z}| \geq 5CN(n)$, we get $\left| \tilde{F}_t'^{(n)}(\tilde{z}) \right| > 1$, i.e., $\tilde{I}'^{(n)}(\tilde{z}) \cap \tilde{\gamma} = \emptyset$.

Recall that the open disks \tilde{V} and \tilde{W} that contain ∞ in Section 5.1. Here, we set $\tilde{V} = \{\tilde{z} \in \tilde{M} \mid |\tilde{z}| > 5CN(n)\}$ and choose an open disk \tilde{W} containing ∞ such that $\tilde{\gamma} \cap \tilde{W} = \emptyset$, and for every $\tilde{z} \in \tilde{V}$, we have $\tilde{I}'^{(n)}(\tilde{z}) \subset \tilde{W}$. Without loss of generality, we can suppose that $z_* \notin \pi(\tilde{\gamma})$. Choose an open disk U containing z_* such that $U \cap \pi(\tilde{\gamma}) = \emptyset$. Write

respectively $\tau(n, z)$ and $\Phi_n(z)$ for the first return time and the first return map of F^n throughout this proof. For every $m \geq 1$, recall that $\tau_m(n, z) = \sum_{i=0}^{m-1} \tau(n, \Phi_n^i(z))$. Let us consider the following value

$$L_m(\tilde{F}^n; 0, 1, z_*) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{F}^n(z_*)}^m.$$

By the same arguments with Lemma 5.4 and Lemma 5.6, we can find multi-paths $\tilde{\Gamma}_i^m(z_*)$ ($1 \leq i \leq P_m(z_*)$) from \tilde{V} to \tilde{V} (see the equations 5.1.1–5.1.5 for the details) such that

$$L_m(\tilde{F}^n; 0, 1, z_*) = \tilde{\gamma} \wedge \prod_{1 \leq i \leq P_m(z_*)} \tilde{\Gamma}_i^m(z_*).$$

For every $j \in \{1, \dots, s\}$ and $(\tilde{z}, \tilde{z}') \in \widetilde{M} \times \widetilde{M} \setminus \widetilde{\Delta}$, there is a unique function $\theta_j : [0, 1] \rightarrow \mathbb{R}$ such that $\theta_j(0) = 0$ and

$$e^{2i\pi\theta_j(t)} = \frac{\tilde{F}_{j,t}(\tilde{z}) - \tilde{F}_{j,t}(\tilde{z}')}{\left| \tilde{F}_{j,t}(\tilde{z}) - \tilde{F}_{j,t}(\tilde{z}') \right|}.$$

Let $\lambda_j(\tilde{z}, \tilde{z}') = \theta_j(1)$. As $\tilde{I}_j \subset \text{Diff}^1(\widetilde{M})$, there is a natural compactification of $\widetilde{M} \times \widetilde{M} \setminus \widetilde{\Delta}$ obtained by replacing the diagonal $\widetilde{\Delta}$ with the unit tangent bundle such that the map λ_j extends continuously (see, for example, [CFGL, page 81]).

Let

$$C_1 = \max_{i \in \{1, \dots, s\}} \sup_{(\tilde{z}, \tilde{z}') \in \widetilde{M} \times \widetilde{M} \setminus \widetilde{\Delta}} |\lambda_j(\tilde{z}, \tilde{z}')|.$$

Suppose that $\widetilde{M}_0 \subset \widetilde{M}$ is a closed fundamental domain with regard to the transformation group G . Denote by $\tilde{I}_i^\pm(\widetilde{M}_0)$ the set $\{\tilde{F}_{i,\pm t}(\tilde{z}) \mid (\pm t, \tilde{z}) \in [0, 1] \times \widetilde{M}_0\}$ where $\tilde{F}_{i,-t} = \tilde{F}_{i,1-t} \circ \tilde{F}_{i,1}^{-1}$. As $\tilde{F}_{i,\pm t} \circ \alpha = \alpha \circ \tilde{F}_{i,\pm t}$ for all $\alpha \in G$ and $t \in [0, 1]$, we can suppose that

$$C_2 = \max_{i \in \{1, \dots, s\}, z \in M} \#\{\tilde{z} \in \pi^{-1}(z) \mid \tilde{I}_i(\tilde{z}) \cap \tilde{I}_i^+(\widetilde{M}_0) \cup \tilde{I}_i^{-1}(\tilde{z}) \cap \tilde{I}_i^-(\widetilde{M}_0) \neq \emptyset\},$$

which is independent of n .

For every $0 \leq j \leq m-1$, $\tau_j(n, z_*) \leq l < \tau_{j+1}(n, z_*)$ and $1 \leq k \leq N(n)$, let $F_{i_0,1}^{\epsilon_{i_0}} = \text{Id}_M$, $\tilde{F}_{i_0,1}^{\epsilon_{i_0}} = \text{Id}_{\widetilde{M}}$,

$$z_{j,l,k} = F_{i_{k-1},1}^{\epsilon_{i_{k-1}}} (F_{i_{k-2},1}^{\epsilon_{i_{k-2}}} \circ \dots \circ F_{i_1,1}^{\epsilon_{i_1}} (F^{n(l-\tau_j(n,z_*))}(\Phi_n^j(z_*))))$$

and

$$\tilde{z}_k^0 = \tilde{F}_{i_{k-1},1}^{\epsilon_{i_{k-1}}} (\tilde{F}_{i_{k-2},1}^{\epsilon_{i_{k-2}}} \circ \dots \circ \tilde{F}_{i_1,1}^{\epsilon_{i_1}}(0)), \quad \tilde{z}_k^1 = \tilde{F}_{i_{k-1},1}^{\epsilon_{i_{k-1}}} (\tilde{F}_{i_{k-2},1}^{\epsilon_{i_{k-2}}} \circ \dots \circ \tilde{F}_{i_1,1}^{\epsilon_{i_1}}(1)).$$

When $\frac{k-1}{N(n)} \leq t \leq \frac{k}{N(n)}$, recall that

$$\tilde{F}_t'^{(n)}(\tilde{z}) = \frac{\tilde{F}_{i_k,N(n)t-(k-1)}^{\epsilon_{i_k}}(\tilde{z}) - \tilde{F}_{i_k,N(n)t-(k-1)}^{\epsilon_{i_k}}(\tilde{z}_k^0)}{\tilde{F}_{i_k,N(n)t-(k-1)}^{\epsilon_{i_k}}(\tilde{z}_k^1) - \tilde{F}_{i_k,N(n)t-(k-1)}^{\epsilon_{i_k}}(\tilde{z}_k^0)}.$$

For every $\tilde{z} \in \tilde{M}$, denote by $\tilde{J}_k(\tilde{z})$ by the curve

$$\tilde{J}_k(\tilde{z}) = \left(\tilde{F}_t'^{(n)}(\tilde{z}) \right)_{\frac{k-1}{N(n)} \leq t \leq \frac{k}{N(n)}}.$$

For every k , define the immersed square

$$\begin{aligned} A_k : [0, 1]^2 &\rightarrow \tilde{M} \\ (t, r) &\mapsto \tilde{F}_{i_k, t}^{\epsilon_{i_k}}(\tilde{z}_k^0) + r(\tilde{F}_{i_k, t}^{\epsilon_{i_k}}(\tilde{z}_k^1) - \tilde{F}_{i_k, t}^{\epsilon_{i_k}}(\tilde{z}_k^0)). \end{aligned}$$

By Equality 7.2.2, we know that $\tilde{J}_k(\tilde{z}) \cap \tilde{\gamma} \neq \emptyset$ implies $\tilde{I}_{i_k}^{\epsilon_{i_k}}(\tilde{z}) \cap A_k \neq \emptyset$ (see Figure 3). Remark here that there are two universal covers in Figure 3, that the curves in \tilde{M} (the big one) is generated by the isotopy $\tilde{I}^{(n)}$, and that the curve in \tilde{M} (the small one) is generated by \tilde{J}_k (and hence by the isotopy $\tilde{I}^{(n)}$ defined by Formular 7.2.1).

Let

$$C_{j,l,k} = \{\tilde{z}_{j,l,k} \in \pi^{-1}(z_{j,l,k}) \mid \tilde{I}_{i_k}^{\epsilon_{i_k}}(\tilde{z}_{j,l,k}) \cap A_k \neq \emptyset\}.$$

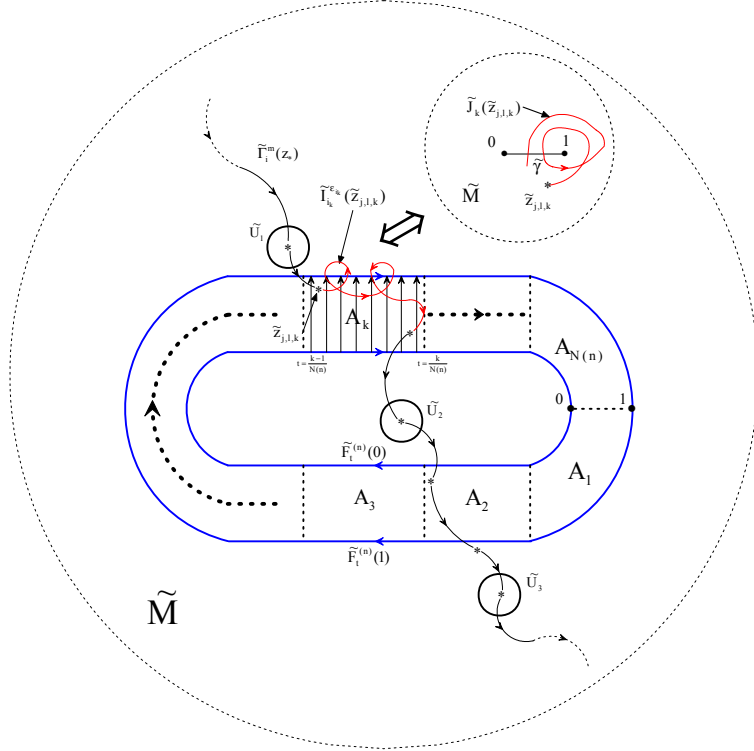


FIGURE 3. The proof of Lemma 7.2

For every k , observing that $\tilde{d}(\tilde{z}_k^0, \tilde{z}_k^1) \leq 2C(k-1) + 1$, there exists $C_3 > 0$ (depending only on \tilde{a} and \tilde{b}) such that

$$(7.2.4) \quad \#\{\alpha \in G \mid A_k \cap \alpha(\tilde{M}_0) \neq \emptyset\} \leq C_3 N(n).$$

Therefore,

$$(7.2.5) \quad \sum_{j,l,k} \#C_{j,l,k} \leq C_2 C_3 N^2(n) \tau_m(n, z_*).$$

We have

$$(7.2.6) \quad L_m(\tilde{F}^n; 0, 1, z_*) = \tilde{\gamma} \wedge \prod_{1 \leq i \leq P_m(z_*)} \tilde{\Gamma}_i^m(z_*) = \tilde{\gamma} \wedge \prod_{j,l,k} \left(\prod_{\tilde{z} \in C_{j,l,k}} \tilde{J}_k(\tilde{z}) \right).$$

We get

$$(7.2.7) \quad \left| L_m(\tilde{F}^n; 0, 1, z_*) \right| \leq c_0 N^2(n) \tau_m(n, z_*),$$

where $c_0 = C_1 C_2 C_3$. Therefore,

$$\left| ni(\tilde{F}; 0, 1, z_*) \right| = \left| i(\tilde{F}^n; 0, 1, z_*) \right| \leq c_0 N^2(n).$$

This implies that, for every $n \geq 1$,

$$0 < \left| i(\tilde{F}; 0, 1, z_*) \right| \leq c_0 \frac{N^2(n)}{n},$$

That is

$$\|F^n\|_{\mathcal{G}} \succeq \sqrt{n}.$$

We have completed this proof. \square

By Proposition 7.2 and Proposition 0.2, we can get the following result which is a generalization of Theorem 1.6 B in [P1] on the closed surface.

Corollary 7.3. *If $F \in \text{Diff}_*^1(M_g, \mu) \setminus \{\text{Id}_{M_g}\}$ ($g > 1$) or $F \in \text{Ham}^1(\mathbb{T}^2, \mu) \setminus \{\text{Id}_{\mathbb{T}^2}\}$, then for any finitely generated subgroup $F \in \mathcal{G} \subset \text{Diff}_*^1(M_g, \mu)$ ($g \geq 1$),*

$$\|F^n\|_{\mathcal{G}} \succeq \sqrt{n}.$$

Moreover, we can improve Corollary 7.3. The following result is our main theorem in this subsection.

Theorem 0.7 *Let $F \in \text{Diff}_*^1(M_g, \mu) \setminus \{\text{Id}_{M_g}\}$ ($g > 1$) (resp. $F \in \text{Ham}^1(\mathbb{T}^2, \mu) \setminus \{\text{Id}_{\mathbb{T}^2}\}$), and $\mathcal{G} \subset \text{Diff}_*^1(M_g, \mu)$ ($g > 1$) (resp. $\mathcal{G} \subset \text{Ham}^1(\mathbb{T}^2, \mu)$) be a finitely generated subgroup containing F , then*

$$\|F^n\|_{\mathcal{G}} \sim n.$$

As a consequence, the groups $\text{Diff}_^1(M_g, \mu)$ ($g > 1$) and $\text{Ham}^1(\mathbb{T}^2, \mu)$ have no distortions.*

The theorem 0.7 can be obtained immediately from the following two lemmas.

Lemma 7.4. *If $F \in \text{Homeo}_*(M_g, \mu) \setminus \text{Hameo}(M_g, \mu)$ ($g > 1$), then for any finitely generated subgroup $F \in \mathcal{G} \subset \text{Homeo}_*(M_g, \mu)$, we have*

$$\|F^n\|_{\mathcal{G}} \sim n.$$

Lemma 7.5. *If $F \in \text{Ham}^1(M_g, \mu) \setminus \{\text{Id}_{M_g}\}$ ($g \geq 1$), then for any finitely generated subgroup $F \in \mathcal{G} \subset \text{Diff}_*^1(M_g, \mu)$, we have*

$$\|F^n\|_{\mathcal{G}} \sim n.$$

Proof of Lemma 7.4. By the definition of $\text{Hameo}(M_g, \mu)$, we know that $\rho_{M_g, I}(\mu) \neq 0$. Assume that $F \in \mathcal{G} = \langle F_{1,1}, \dots, F_{s,1} \rangle \subset \text{Homeo}_0(M_g, \mu)$ and I_i ($1 \leq i \leq s$) are the identity isotopies corresponding to $F_{i,1}$. Denote by $\|\cdot\|_{H_1(M_g, \mathbb{R})}$ the norm in the space $H_1(M_g, \mathbb{R})$. Write

$$\kappa = \max_{i \in \{1, \dots, s\}} \left\{ \|\rho_{M_g, I_i}(\mu)\|_{H_1(M_g, \mathbb{R})} \right\}.$$

As $\rho_{M_g, I}(\mu) \neq 0$ and $F \in \mathcal{G}$, we have $\kappa > 0$.

For every $n \in \mathbb{N}$, if I^n is homotopic to $\prod_{s=1}^{N(n)} I_{i_s}^{\epsilon_{i_s}}$, then we have

$$n \cdot \|\rho_{M_g, I}(\mu)\|_{H_1(M_g, \mathbb{R})} = \|\rho_{M_g, I^n}(\mu)\|_{H_1(M_g, \mathbb{R})} = \left\| \sum_{s=1}^{N(n)} \rho_{M_g, I_{i_s}^{\epsilon_{i_s}}}(\mu) \right\|_{H_1(M_g, \mathbb{R})} \leq \kappa \cdot N(n).$$

Hence $\|F^n\|_{\mathcal{G}} \succeq n$. On the other hand, we have $\|F^n\|_{\mathcal{G}} \preceq n$, which completes the proof. \square

Proof of Lemma 7.5. For simplicity, we write M_g as M . It is sufficient to prove that $\|F^n\|_{\mathcal{G}} \succeq n$. We use the notations in the proofs of Proposition 0.2 and Lemma 7.2.

If $\sharp \text{Fix}_{\text{Cont}, I}(F) = +\infty$, assume that $X \subset \text{Fix}_{\text{Cont}, I}(F)$, I' , $Y = \{a, b\} \subset X$, I'_Y are the notations defined in the proof for the case $\sharp \text{Fix}_{\text{Cont}, I}(F) = +\infty$ of Proposition 0.2. If $\sharp \text{Fix}_{\text{Cont}, I}(F) < +\infty$, for convenience, we require $a = \alpha(\lambda)$, $b = \omega(\lambda)$, and $I'_Y = I'$ where $\alpha(\lambda)$, $\omega(\lambda)$ and I' are the notions defined in the proof for the case $\sharp \text{Fix}_{\text{Cont}, I}(F) < +\infty$ of Proposition 0.2. Suppose that \tilde{I}' , \tilde{I}'_Y are respectively the lifts of I' and I'_Y to \tilde{M} . Choose a lift \tilde{a} of a and a lift \tilde{b} of b . We know that $I_\mu(\tilde{F}; a, b) \neq 0$. As $F \neq \text{Id}_M$ and μ has total support, by the property (B2) in the proof of Proposition 0.2, we can choose $z_* \in \text{Rec}^+(F) \setminus X$, such that $\rho_{M, I}(z_*)$ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ exist, and $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ is not zero.

Suppose now that $z \in M \setminus X$. By the items (3) and (5) of Theorem 6.10, we know that $I(z)$ and $I'_Y(z)$ are homotopic in M . Hence, for every $n \in \mathbb{N}$, $I^{(n)}(z) = \prod_{j=1}^{N(n)} I_{i_j}^{\epsilon_{i_j}}(z)$ is homotopic to $(I'_Y)^n(z)$ in M . If $\gamma_{F^n(z), z}$ is a geodesic path from $F^n(z)$ and z on M , similarly to the proof of Formula 1.3.1, then there exists $C' > 0$ such that

$$(7.2.8) \quad \left\| [I^{(n)}(z) \gamma_{F^n(z), z}]_M \right\|_{H_1(M, \mathbb{R})} = \left\| [(I'_Y)^n(z) \gamma_{F^n(z), z}]_M \right\|_{H_1(M, \mathbb{R})} \leq C' N(n).$$

Assume that $C_{j,l,k}$, $\tilde{\gamma}$, $\tilde{I}^{(n)}$, C_1 and C_2 are the notations in the proof of Lemma 7.2.

Write

$$C'_{j,l,k} = \{\tilde{z}_{j,l,k} \in \pi^{-1}(z_{j,l,k}) \mid \tilde{I}_{i_k}^{\epsilon_{i_k}}(\tilde{z}_{j,l,k}) \cap (A_k(\{r=0\} \cup \{r=1\})) \neq \emptyset\}.$$

Obviously,

$$\#C'_{j,l,k} \leq 2C_2, \quad \sum_{j,l,k} \#C'_{j,l,k} \leq 2C_2 N(n) \tau_m(n, z_*).$$

Observing that $(\tilde{I}'_Y)^n$ and $\tilde{I}'^{(n)}$ are two isotopies from $\text{Id}_{\tilde{M}}$ to \tilde{F}^n which fix \tilde{a} and \tilde{b} , by Remark 1.4, $(\tilde{I}'_Y)^n$ is homotopic to $\tilde{I}'^{(n)}$ in $\tilde{M} \setminus \{\tilde{a}, \tilde{b}\}$.

Observe that $N(n)$ has the following simple properties:

- For every two numbers $n_1, n_2 \geq 1$, we have $N(n_1 + n_2) \leq N(n_1) + N(n_2)$;
- For every $1 \leq k < n$, we have $N(k) - N(1) \leq N(k+1) \leq N(k) + N(1)$.

By the properties of $N(n)$ listed above and Inequality 7.2.8, we get for every $z \in M \setminus \{a, b\}$, there exists $C'_2 > 0$ such that

$$(7.2.9) \quad \#\{\tilde{z} \in \pi^{-1}(z) \mid \tilde{I}'^{(n)}(\tilde{z}) \wedge \tilde{\gamma} \neq 0\} \leq C'_2 N(n).$$

Write

$$C'_{j,l} = \{\tilde{z} \in \pi^{-1}(z_{j,l,N(n)}) \mid \tilde{I}'^{(n)}(\tilde{z}) \wedge \tilde{\gamma} \neq 0\}.$$

Under the hypotheses of Lemma 7.5, we want to improve the value $N^2(n)$ to $N(n)$ in the inequality 7.2.7.

Based on the analyses above, we have

$$L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}'^{(n)}, z_*}^m = \tilde{\gamma} \wedge \tilde{\Gamma}_{(\tilde{I}'_Y)^n, z_*}^m = \pi(\tilde{\gamma}) \wedge \Gamma_{(I'_Y)^n, z_*}^m.$$

To estimate the value $L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*)$, we need to consider the isotopy $\tilde{I}'^{(n)}$ (If we use the isotopy $(\tilde{I}'_Y)^n$, the difficulty is we do not know how the isotopy $(I'_Y)^n$ rotates around the points $\pi(\tilde{a})$ and $\pi(\tilde{b})$). If the immersed squares A_k are uniformly bounded for n , then by the inequalities 7.2.4–7.2.7, we have done. We explain now the case where the squares A_k are not bounded for n is also true. The inequality 7.2.9 shows that, for every fixed j and l , it is sufficient to consider at most $C'_2 N(n)$ elements of $C'_{j,l}$. For every $\tilde{z} \in C'_{j,l}$, we consider the value $V(\tilde{z}) = |\tilde{\gamma} \wedge \tilde{I}'^{(n)}(\tilde{z})|$ (Maybe it is no sense since the ends of $\tilde{I}'^{(n)}(\tilde{z})$ are possibly on $\tilde{\gamma}$. In this case, we let $V(\tilde{z}) = |\tilde{\gamma} \wedge \text{Int}(\tilde{I}'^{(n)}(\tilde{z}))| + 2$). We can write $\tilde{I}'^{(n)}(\tilde{z})$ as the concatenation of $N(n)$ sub-paths $\tilde{J}_k(\tilde{z})$ ($k = 1, \dots, N(n)$). Obviously, we have

$$\left| L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) \right| \leq \left| \tilde{\gamma} \wedge \prod_{j,l,k} \left(\prod_{\tilde{z} \in C'_{j,l,k}} \tilde{J}_k(\tilde{z}) \right) \right| + \left| \tilde{\gamma} \wedge \prod_{j,l} \left(\prod_{\tilde{z} \in C'_{j,l}} \tilde{I}'^{(n)}(\tilde{z}) \right) \right|.$$

We know that the value of the first part of the right hand of the inequality above is less than $2C_1 C_2 N(n) \tau_m(n, z_*)$. Hence, to explore the relation of the bound of $\left| L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) \right|$ and the power of $N(n)$, we can suppose that the path $\tilde{J}_k(\tilde{z})$ never meets $A_k(\{r=0\} \cup \{r=1\})$ for every k and $\tilde{z} \in C'_{j,l}$. As the isotopies \tilde{I}_i ($1 \leq i \leq s$) commutes with the transformations, we can observe that (see Figure 4 and refer to the proof of Lemma 7.2) if we demand the values $V(\tilde{z})$ for some $\tilde{z} \in C'_{j,l}$ (in all of the probabilities) as large as

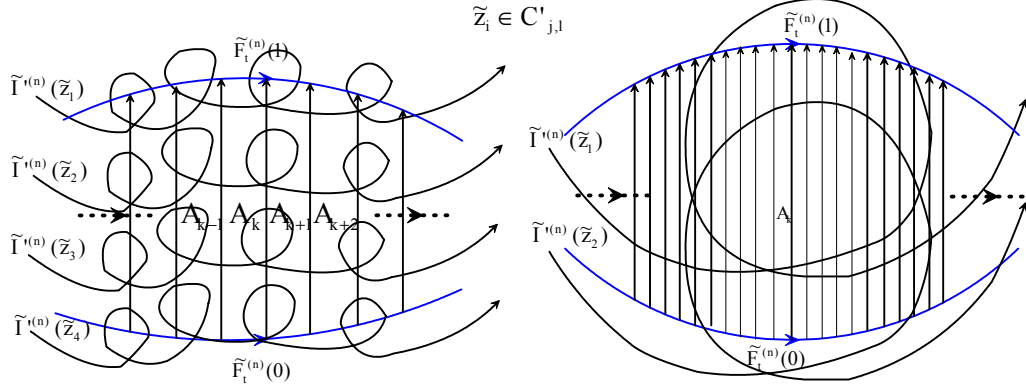


FIGURE 4. The proof of Lemma 7.5

possible, then other values of $V(\tilde{z})$ will be as small as possible; if we demand most of the values $V(\tilde{z})$ increase, then the maximal of $V(\tilde{z})$ will decrease. It is easy to prove that $\sum_{z \in C'_{j,l}} V(\tilde{z}) \leq C'_1 N(n)$ for some $C'_1 > 0$. Hence,

$$\left| \tilde{\gamma} \wedge \left(\prod_{j,l} \prod_{\tilde{z} \in C'_{j,l}} \tilde{I}^{(n)}(\tilde{z}) \right) \right| = \left| \tilde{\gamma} \wedge \left(\prod_{j,l} \prod_{\tilde{z} \in C'_{j,l}} \prod_{k=1}^{N(n)} \tilde{J}_k(\tilde{z}) \right) \right| \leq C'_1 N(n) \tau_m(n, z_*).$$

Therefore, there exists $C''_1 > \{C_1, C'_1\}$ such that

$$\left| L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) \right| \leq C''_1 (2C_2 + C'_2) N(n) \tau_m(n, z_*).$$

It implies that, for every $n \geq 1$,

$$(7.2.10) \quad 0 < |i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)| \leq c'_0 \frac{N(n)}{n},$$

where $c'_0 = C''_1 (2C_2 + C'_2)$. This implies that

$$\|F^n\|_{\mathcal{G}} \succeq n.$$

Therefore, $\|F^n\|_{\mathcal{G}} \sim n$, which completes the proof. \square

As a consequence of Theorem 0.7, we have the following theorem:

Theorem 7.6. *Let \mathcal{G} be a finitely generated group with generators $\{g_1, \dots, g_s\}$ and $f \in \mathcal{G}$ be an element which is distorted with respect to the word norm on \mathcal{G} . Then $\phi(f) = \text{Id}_{\mathbb{T}^2}$ (resp. $\phi'(f) = \text{Id}_{M_g}$ where $g > 1$) for any homomorphism $\phi : \mathcal{G} \rightarrow \text{Ham}^1(\mathbb{T}^2, \mu)$ (resp. $\phi' : \mathcal{G} \rightarrow \text{Diff}_*^1(M_g, \mu)$ with $g > 1$). In particular, if \mathcal{G} is a finitely generated subgroup of $\text{Ham}^1(\mathbb{T}^2, \mu)$ (resp. $\text{Diff}_*^1(M_g, \mu)$ with $g > 1$), every element of $\mathcal{G} \setminus \{\text{Id}_{M_g}\}$ ($g \geq 1$) is undistorted with respect to the word norm on \mathcal{G} .*

Proof. We only prove the case where $\phi : \mathcal{G} \rightarrow \text{Ham}^1(\mathbb{T}^2, \mu)$ since other cases follow similarly. Let \mathcal{G}' be the finitely generated group generated by $\{\phi(g_1), \dots, \phi(g_s)\}$. As f is a distortion element of \mathcal{G} , there exists a subsequence $\{n_i\}_{i \geq 1} \subset \mathbb{N}$ such that

$$\lim_{i \rightarrow +\infty} \frac{\|\phi^{n_i}(f)\|_{\mathcal{G}'}}{n_i} = \lim_{i \rightarrow +\infty} \frac{\|\phi(f^{n_i})\|_{\mathcal{G}'}}{n_i} = 0.$$

By Theorem 0.7, we have $\phi(f) = \text{Id}_{\mathbb{T}^2}$. \square

Let us recall some results about the irreducible lattice $\text{SL}(n, \mathbb{Z})$ with $n \geq 3$. The lattice $\text{SL}(n, \mathbb{Z})$ and its any normal subgroup of finite order have the following properties:

- It contains a subgroup isomorphic to the group of upper triangular integer valued matrices of order 3 with 1 on the diagonal (the integer Heisenberg group), which tells us the existence of distortion element of every infinite norm subgroup of $\text{SL}(n, \mathbb{Z})$ (see [P2, Prop. 1.7]);
- It is almost simple (every normal subgroup is finite or has a finite index) which is due to Margulis (Margulis finiteness theorem, see [Mar]).

Applying these results above and Theorem 7.6, we get the following result:

Theorem 0.8 *Every homomorphism from $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) to $\text{Ham}^1(\mathbb{T}^2, \mu)$ or $\text{Diff}_*^1(M_g, \mu)$ ($g > 1$) is trivial.*

Proof. Again, we only prove the case where $\phi : \mathcal{G} \rightarrow \text{Ham}^1(\mathbb{T}^2, \mu)$ since other cases follow similarly. The following argument is due to Polterovich [P2, Proof of Theorem 1.6]. By the first item of properties of $\text{SL}(n, \mathbb{Z})$, there is a distortion element f in $\text{SL}(n, \mathbb{Z})$. Apply Theorem 7.6 to the distortion element f of infinite order of $\text{SL}(n, \mathbb{Z})$. We have that f lies in the kernel of ϕ . Note that $\text{Ker}(\phi)$ is an infinite normal subgroup of $\text{SL}(n, \mathbb{Z})$. By the second item of properties of $\text{SL}(n, \mathbb{Z})$, $\text{Ker}(\phi)$ has finite index in $\text{SL}(n, \mathbb{Z})$. Hence the quotient $\text{SL}(n, \mathbb{Z})/\text{Ker}(\phi)$ is finite. Therefore, ϕ has finite images. Applying Corollary 0.6, we get ϕ is trivial. \square

Finally, let us recall a classical result about the mapping class group $\text{Mod}(M)$, where $\text{Mod}(M) = \text{Homeo}^+(M)/\text{Homeo}_*(M)$ is the isotopy classes of orientation-preserving homeomorphisms of M (see [FM]): any homomorphism $\phi : \Gamma \rightarrow \text{Mod}(M)$ has finite images where Γ is an irreducible lattice in a semisimple lie group of \mathbb{R} -rank at least two.

Applying the result above and Theorem 0.8, we get the following general conjecture of Zimmer in the special case of surfaces:

Theorem 0.9 *Every homomorphism from $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) to $\text{Diff}^1(M_g, \mu)$ ($g > 1$) has only finite images.*

8. APPENDIX

Appendix A.

Lemma 8.1. *Let (X, d) be a metric space and $f : X \rightarrow X$ be a continuous map. A positively recurrent point of f is also a positively recurrent point of f^q for all $q \in \mathbb{N}$.*

Proof. If $z \in \text{Rec}^+(f)$, let $O_i = \{z' \in X \mid d(z, z') < \frac{1}{i}\}$ for $i \in \mathbb{N} \setminus \{0\}$. We suppose that $f^{n_k}(z) \rightarrow z$ when $k \rightarrow +\infty$. Write $n_k = l_k q + p_k$ where $0 \leq p_k < q$. If there are infinitely many k such that $p_k = 0$, we are done. Otherwise, there are infinitely many k such that $p_k = p$ where $0 < p < q$. We can suppose that $f^{l_k q + p}(z) \rightarrow z$ when $k \rightarrow +\infty$ by considering subsequence if necessary. We suppose that $f^{l_{k_1} q + p}(z) \in O_{m_1}$, then there exists O_{m_2} such that $f^{l_{k_1} q + p}(O_{m_2}) \subset O_{m_1}$. Similarly, there exists l_{k_2} and O_{m_3} such that $f^{l_{k_1} q + p}(O_{m_3}) \subset O_{m_2}$. By induction, there is a subsequence $(l_{k_j})_{j \geq 1}$ of $(l_k)_{k \geq 1}$ and a subsequence $\{O_{m_j}\}_{j \geq 1}$ of $\{O_m\}_{m \geq 1}$ such that $f^{l_{k_j} q + p}(O_{m_{j+1}}) \subset O_{m_j}$. Consider the subsequence $\{f^{q(p + \sum_{j=(t-1)q}^{tq-1} l_{k_j})}(z)\}_{t \geq 1}$, we are done. \square

Appendix B.

We fix a closed surface M of genus $g \geq 1$ and a topological closed disk D on M all examples will coincide with the identity outside of D including isotopies. Up to a diffeomorphism, we may suppose that D is the closed unit Euclidean disk. We will construct an identity isotopy $I = (F_t)_{t \in [0,1]}$, we will write $F = F_1$ and $\tilde{F} = \tilde{F}_1$ the time-one map of $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]}$ that is the lifted identity isotopy of I on the universal covering space $\pi : \tilde{M} \rightarrow M$.

Example 8.2. We construct an isotopy I of M and a measure $\mu \in \mathcal{M}(F)$ such that

- $F \notin \text{Diff}(M)$;
- I satisfies the B-property;
- there are two different fixed points \tilde{z}_0 and \tilde{z}_1 of \tilde{F} such that the linking number $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z)$ is not bounded;
- there are two different fixed points \tilde{z}_0 and \tilde{z}_1 of \tilde{F} such that the linking number $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z)$ is not μ -integrable.

Use the polar coordinate for D with the center $z_0 = (0, 0)$ and suppose $z_1 = (4/5, 0)$. Let $D_{p/q} = \{(r, \theta) \mid r \in]0, p/q[\}$ where $p/q \in]0, 1[\cap \mathbb{Q}$.

Consider a smooth decreasing function $\alpha : [0, 3/4] \rightarrow \mathbb{R}$ such that $\alpha|_{[0, 1/2]} \equiv 1$ and $\alpha = 0$ on neighborhood of $3/4$.

Consider a C^∞ -diffeomorphism $\rho(r)$ of $]0, 3/4[$ as follows

- $\rho(r)$ fixes the point $1/k$ for every $k > 1$ and $\rho(r) = r$ when $r \in [1/2, 3/4[$;
- $\rho^n(r) \rightarrow 1/(k+1)$ when $n \rightarrow -\infty$ for every $k > 1$ and $r \in]1/(k+1), 1/k[$;
- $\rho^n(r) \rightarrow 1/k$ when $n \rightarrow +\infty$ for every $k > 1$ and $r \in]1/(k+1), 1/k[$.

Consider the following homeomorphism F of D defined on D by the formula:

$$(8.0.11) \quad F(re^{2i\pi\theta}) = \begin{cases} \rho(r)e^{2i\pi(\theta + \alpha(r)(2^{\frac{1}{r}} + \frac{1}{2}))} & \text{on } D_{3/4}; \\ \text{Id} & \text{on } D \setminus D_{3/4}. \end{cases}$$

We construct an isotopy $I = (F_t)_{t \in [0,1]}$ on D by replacing $\alpha(r)(2^{\frac{1}{r}} + \frac{1}{2})$ with $t\alpha(r)(2^{\frac{1}{r}} + \frac{1}{2})$ and $\rho(r)$ with $(1-t)r + t\rho(r)$ in Formula (8.0.11). It is easy to see that \tilde{F} is not differentiable at z_0 .

Consider a finite measure μ on M that is invariant by F as follows

$$\mu = \sum_{k \geq 2} 2^{-(k-1)} \mu_k$$

where μ_k is the Lebesgue probability measure on C_k .

Let $B_k = \{(r, \theta) \mid r \in]1/(k+1), 1/k[\}$ and $C_k = \{z \in D \mid |z| = 1/k\}$ ($k \geq 2$). Fix one point $z_k \in C_k$ for every $k \geq 2$. Let \tilde{z}_k ($k \geq 0$) be any lift of z_k contained in a connected component of $\pi^{-1}(D)$. For any point $z \in B_k$, the ω -limit set of z is included in C_k and the α -limit set of z is included in C_{k+1} . When $z \in C_k$, the angle of the trajectory of $I(z)$ rotating around z_0 is $(2^{k+1} + 1)\pi$. Hence F has no contractible fixed points on $D_{1/2}$. When $z \in D_{3/4} \setminus D_{1/2}$, the angle of the trajectory of $I(z)$ rotating around z_0 is uniformly bounded. Therefore, I satisfies the B-property. However, $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z_k) = 2^k + 1/2$ and $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z)$ is not μ -integrable. Remark that the support of μ is not the whole space.

Example 8.3. We construct an isotopy I of M and a measure $\mu \in \mathcal{M}(F)$ with total support and no atoms on $\text{Fix}_{\text{Cont}, I}(F)$ such that

- $\rho_{M, I}(\mu) = 0$;
- $F \in \text{Diff}(M)$ (and hence I satisfies the WB-property);
- I does not satisfy the B-property (and hence $F \notin \text{Diff}_*^1(M)$);
- there is a compact set $\tilde{P} \subset \tilde{M}$ and $\{(\tilde{z}_k, \tilde{z}'_k)\}_{k \geq 1} \subset \text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F}) \setminus \tilde{\Delta}$ in $\tilde{P} \times \tilde{P}$, such that the linking numbers $i(\tilde{F}; \tilde{z}_k, \tilde{z}'_k, z)$ are not uniformly bounded;
- the action L_μ (see 6.1) is not bounded;
- the action L_μ and l_μ are not continuous.

Use the Cartesian (x, y) -coordinate system in D and suppose $z_0 = (0, 0)$. On the x -axis, we suppose that B_k ($k \geq 1$) is a ball whose center is on $z_k = 1/(k+1) + 1/(2k(k+1))$ and whose radius is $r_k = 1/2(k+1)^2$.

Consider a family of smooth functions $\alpha_k : [0, r_k] \rightarrow \mathbb{R}$ such that $\alpha_k = 0$ on neighborhoods of 0 and r_k , $\alpha_k(r_k/2) = 2(-1)^k(k+1)^5$ and

$$2\pi \int_0^{r_k} \alpha_k(r) r \, dr = (-1)^k k.$$

Consider the following diffeomorphism F of D which is defined by the formula:

$$(8.0.12) \quad F(z_k + re^{2i\pi\theta}) = \begin{cases} z_k + re^{2i\pi(\theta + \alpha_k(r))} & \text{on } B_k; \\ \text{Id} & \text{on } D \setminus \bigcup_{k \geq 1} B_k. \end{cases}$$

We construct an isotopy $I = (F_t)_{t \in [0, 1]}$ on D by replacing $\alpha_k(r)$ with $t\alpha_k(r)$ in Formula (8.0.12).

Obviously, z_k and $z'_k = z_k + r_k/2$ are fixed points of F and we have

$$i(\tilde{F}; \tilde{z}_k, z'_k) = 2(-1)^k(k+1)^5$$

and

$$i(\tilde{F}; \tilde{z}_0, \tilde{z}_k, z'_k) = \rho_{A_{\tilde{z}_0, \tilde{z}_k}, \tilde{F}_{\tilde{z}_0, \tilde{z}_k}}(\tilde{z}_k') = 2(-1)^{k+1}(k+1)^5$$

where \tilde{z}_0, \tilde{z}_k and \tilde{z}_k' are contained in a connected component \tilde{D} of $\pi^{-1}(D)$. Hence I does not satisfy the B-property and there is a compact set $\text{Cl}(\tilde{D})$ and $\{\tilde{z}_k\}_{k \geq 1} \subset \text{Fix}(F) \setminus \{\tilde{z}_0\}$ in $\text{Cl}(\tilde{D})$, such that the linking numbers $i(\tilde{F}; \tilde{z}_0, \tilde{z}_k, z)$ are not uniformly bounded.

It is easy to prove that F is a diffeomorphism of M but it is not a C^1 -diffeomorphism of M : its differential DF is not continuous at z_0 .

Consider a finite measure μ on M satisfying that

- μ has total support;
- μ is non-atomic;
- μ restricted on B_k is the Lebesgue measure with $\mu(B_k) = \pi r_k^2$ for every $k \geq 1$.

Obviously, $\mu \in \mathcal{M}(F)$ and $\rho_{M,I}(\mu) = 0$. Furthermore, we have

$$I_\mu(\tilde{F}; z_{k+1}, z_k) = i_\mu(\tilde{F}; \tilde{z}_{k+1}, \tilde{z}_k) = (-1)^{k+1}(2k+1)$$

and

$$I_\mu(\tilde{F}; z_0, z_k) = i_\mu(\tilde{F}; \tilde{z}_0, \tilde{z}_k) = (-1)^{k+1}k.$$

Therefore, the action L_μ is not bounded. Observe that $z_k \rightarrow z_0$ and $\tilde{z}_k \rightarrow \tilde{z}_0$ as $k \rightarrow +\infty$, so that L_μ and l_μ are not continuous (at z_0 and \tilde{z}_0).

REFERENCES

- [BLFM] F. BÉGUIN; P. LE CALVEZ, S. FIRMO AND T. MIERNOWSKI : Des points fixes communs pour des difféomorphismes de \mathbf{S}^2 qui commutent et préservent une mesure de probabilité, In preparation.
- [CFGL] S. CROVISIER; J. FRANKS; J.M., GAMBAUDO; P., LE CALVEZ : Dynamique des difféomorphismes conservatifs des surfaces: un point de vue topologique. [Dynamics of conservative surface diffeomorphisms: a topological viewpoint], Panoramas et Synthèses [Panoramas and Syntheses], **21** *Société Mathématique de France, Paris*, 2006.
- [Fa] A. FATHI : Structure of the group of homeomorphisms preserving a good measure on a compact manifold, *Ann. Sci. École Norm. Sup. (4)*, **13** (1980), no.1, 285-299.
- [Fi] D. FISHER : Groups acting on manifolds: around the Zimmer program, *Geometry, rigidity, and group actions*, 72-157, Chicago Lectures in Math., *Univ. Chicago Press, Chicago, IL*, 2011.
- [Fr1] J. FRANKS : Generalizations of the Poincaré-Birkhoff theorem, *Annals of Math. (2)*, **128** (1988), 139-151; Erratum to: "Generalizations of the Poincar-Birkhoff theorem" [Ann. of Math. (2) **128** (1988), 139-151], *Annals of Math. (2)*, **164** (2006), 1097-1098.
- [Fr2] J. FRANKS : Rotation vectors and fixed points of area preserving surface diffeomorphisms, *Trans. Amer. Math. Soc.*, **348** (1996), 2637-2662.
- [FH1] J. FRANKS AND M. HANDEL : Periodic points of Hamiltonian surface diffeomorphisms, *Geom. Topol.*, **7** (2003), 713-756.
- [FH2] J. FRANKS AND M. HANDEL : Distortion Element in Group actions on surface, *Duke Math. J.*, **131** (2006), 441-468.
- [FM] B. FARB AND H. MASUR : Superrigidity and mapping class groups, *Topology*, no. 6, **37** (1998), 1169-1176.
- [Ham1] M.-E. HAMSTROM : The space of homeomorphisms on a torus, *Illinois J. Math.*, vol.**9** (1965), 59-65.
- [Ham2] M.-E. HAMSTROM : Homotopy groups of the space of homeomorphisms on a 2-manifold, *Illinois J. Math.*, vol.**10** (1966), 563-573.
- [Han] M. HANDEL : Commuting homeomorphisms of S^2 , *Topology*, **31** (1992), no. 2, 293-303.
- [J] O. JAULENT : Existence d'un feuilletage positivement transverse à un homéomorphisme de surface, In preparation.

- [Ka] M. KAC : On the notion of recurrence in discrete stochastic processes, *Bull. Am. Math. Soc.*, **53** (1947), 1002-1010.
- [Kn] H. KNESER : Die Deformationssätze der einfach zusammenhängenden Flächen, *Math. Z.*, **25** (1926), 362-372.
- [Lec1] P. LE CALVEZ : Une version feuilletée équivariante du théorème de translation de Brouwer, *Inst. Hautes Études Sci. Publ. Math.*, **102** (2005), 1-98.
- [Lec2] P. LE CALVEZ : Periodic orbits of Hamiltonian Homeomorphisms of surfaces, *Duke Math. J.*, **133** (2006), 126-184.
- [Ler] F. LE ROUX : Étude topologique de l'espace des homéomorphismes de Brouwer (I), *Topology*, **40** (2001).
- [Mar] G.A. MARGULIS : Discrete Subgroups of Semisimple Lie Groups, *Ergeb. Math. Grenzgeb*, Berlin, Springer, 1991, **17**, no. 3.
- [Mat] S. MATSUMOTO : Arnold conjecture for surface homeomorphisms, *Topol. Appl.*, **104** (2000), 191-214.
- [MS] D. McDUFF AND D. SALAMON : Introduction to symplectic topology, Oxford Mathematical Monographs, *Oxford University Press*, 1995.
- [N] T. NEEDHAM : Visual Complex Analysis, *Oxford University Press*, 1997.
- [P1] L. POLTEROVICH : Growth of maps, distortion in groups and symplectic geometry, *Invent. Math.*, **150(3)** (2002), 655-686.
- [P2] L. POLTEROVICH : Floer homology, dynamics and groups. *Morse theoretic methods in nonlinear analysis and in symplectic topology*, 417-438, *NATO Sci. Ser. II Math. Phys. Chem.* **217**, Springer, Dordrecht, 2006.
- [St] S. SCHWARTZMAN : Asymptotic cycles, *Ann. of Math. (2)*, **68** (1957), 270-284.
- [Sz] M. SCHWARZ : On the action spectrum for closed symplectically aspherical manifolds, *Pacific J. Math.*, **193** (2000), 419-461.
- [Z] R. ZIMMER : Actions of semisimple groups and discrete subgroups, *Proceedings of the International Congress of Mathematicians*, Vol. 1,2, Berkeley, Calif., 1986: 1247-1258, *Amer. Math. Soc.*, Providence, RI, 1987.

CHERN INSTITUTE OF MATHEMATICS, NANKAI UNIVERSITY, TIANJIN 300071, P.R.CHINA.
 E-mail address: wangjian@nankai.edu.cn